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Finite-dimensional attractors for the quasi-linear strongly-damped wave equation

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ABSTRACT

We present a new method of investigating the so-called quasi-linear strongly-damped wave equations

$$\partial_t^2 u - \gamma \partial_t \Delta_x u - \Delta_x u + f(u) = \nabla_x \cdot \phi'(\nabla_x u) + g$$

in bounded 3D domains. This method allows us to establish the existence and uniqueness of energy solutions in the case where the growth exponent of the non-linearity ϕ is less than 6 and f may have arbitrary polynomial growth rate. Moreover, the existence of a finite-dimensional global and exponential attractors for the solution semigroup associated with that equation and their additional regularity are also established. In a particular case $\phi \equiv 0$ which corresponds to the so-called semi-linear strongly-damped wave equation, our result allows to remove the long-standing growth restriction $|f(u)| \leqslant C(1 + |u|^5)$.

1. Introduction

We consider the following quasi-linear strongly-damped wave equation in a smooth bounded domain $\Omega \subset \mathbb{R}^3$:

$$\begin{cases} \partial_t^2 u - \gamma \partial_t \Delta_x u - \Delta_x u + f(u) = \nabla_x \cdot \phi'(\nabla_x u) + g, \\ u|_{\partial\Omega} = 0, \quad \xi_u(0) := \big(u(0), \partial_t u(0)\big) = \xi_0, \end{cases} \tag{1.1}$$

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where Δ_x is a Laplacian with respect to the variable $x \in \Omega$, $u = u(t,x)$ is an unknown function, $\gamma > 0$ is a fixed positive number, $g \in L^2(\Omega)$ are given external forces, and $\phi \in C^2(\mathbb{R}^3 \to \mathbb{R}^1)$ and $f \in C^2(\mathbb{R}^1 \to \mathbb{R}^1)$ are known functions satisfying the following conditions:

$$a_0|\eta|^{p-1} \leqslant \phi''(\eta) \leqslant a_1\big(1 + |\eta|^{p-1}\big), \quad \forall \eta \in \mathbb{R}^3, \tag{1.2}$$

for some positive a_i and $p \in [1,5)$ and

$$-C + a|s|^q \leqslant f'(s) \leqslant C\big(1 + |s|^q\big), \quad \forall s \in \mathbb{R}^1, \tag{1.3}$$

for some $C > 0$, $a > 0$, and $q > 0$. Note that assumption (1.2) reads

$$a_0|\eta|^{p-1}|\xi|^2 \leqslant \sum_{i,j=1}^{3} \phi''_{\eta_i\eta_j}(\eta)\xi_i\xi_j \leqslant a_1\big(1 + |\eta|^{p-1}\big)|\xi|^2, \quad \forall \xi, \eta \in \mathbb{R}^3, \tag{1.4}$$

and typical examples of the non-linearities satisfying the above conditions (1.2) and (1.3) are

$$f(u) = u|u|^q - Cu, \qquad \phi(\eta) = |\eta|^{p+1}. \tag{1.5}$$

Equations of the form (1.1) occur in the study of motion of viscoelastic materials. For instance, in one-dimensional case, they model longitudinal vibration of a uniform, homogeneous bar with non-linear stress law given by the function $\phi(u_x)$. In two- and three-dimensional cases they describe antiplane shear motions of viscoelastic solids. We refer to [28,41,44] for physical origins and derivation of mathematical models of motions of viscoelastic media and only recall here that, in applications, the unknown u naturally represents the displacement of the body relative to a fixed reference configuration. By this reason, it would be more physical to consider Eq. (1.1) in the *vector* case $u = (u^1, u^2, u^3)$. However, in order to avoid the additional technicalities, we prefer to study only the *scalar* case of Eq. (1.1) although the most part of our results can be straightforwardly extended to the vector case with the *convex* energy ϕ satisfying the analogue of conditions (1.2). It is also worth to note that, even for $n = 1$, problem (1.1) may not have a global classical solution if the viscosity term $\gamma\partial_t\Delta_x u$ is omitted. Thus the inclusion of this damping term represents a regularization of the equation and prevents the blow-up of solutions.

The Cauchy problem (for the case $\Omega = \mathbb{R}^n$) and the boundary value problem for Eq. (1.1) under the different assumptions on the non-linearities ϕ and f have been studied in many papers (see [2,9,15–17,23–25,27,30,31,34,35,38,39,42,44,46,47,51,55,57,58] and references therein). The most understood is the semi-linear case $\phi \equiv 0$:

$$\partial_t^2 u - \gamma\partial_t\Delta_x u - \Delta_x u + f(u) = g \tag{1.6}$$

which is usually refereed as strongly-damped wave equation or pseudo-hyperbolic equation, see, e.g., [4,32,34,36,37,42,50,54]. As well as we know, one of the first essential results on global behavior of solutions problem (1.6) was obtained by Webb [54]. He proved that, if $\Omega \subset \mathbb{R}^3$ and the non-linear term satisfies the standard dissipativity conditions (without any growth restrictions), problem (1.6) has a unique strong solution belonging to the space $[H^2(\Omega) \cap H_0^1(\Omega)] \times L^2(\Omega)$ and each strong solution tends to the appropriate stationary solution as $t \to \infty$. This result inspired the further studies of the long-time behavior of solutions of that equations. In particular, the related operator-differential equation in a Banach space X:

$$u_{tt} + \alpha Au_t + Au = f(u), \quad \alpha > 0, \tag{1.7}$$

was considered in [29,45]. Here A is a sectorial operator with compact resolvent on X and f is a non-linear operator satisfying some regularity and growth conditions. Then the semigroup $S(t)$ generated by the Cauchy problem for (1.7) in the phase space $X^\sigma \times X^\beta$, $0 \leqslant \sigma \leqslant \beta < 1$, $X^s := D(A^{s/2})$ is well-posed, dissipative and asymptotically compact and therefore possesses a global attractor in the above phase space.

Thus, it was known for a long time that Eq. (1.6) is well-posed and dissipative in the class of *strong* solutions (say, in the phase space $[H^2(\Omega) \cap H^1_0(\Omega)] \times H^1_0(\Omega))$ and this result does not require any growth restrictions on the non-linearity f (only the natural quasi-monotonicity assumption $f' \geqslant -K$ should be posed, see, e.g., [37,54]).

However, the situation seemed principally different for the case of less regular *energy* solutions. Indeed, roughly speaking, an energy solution means a solution with a finite energy. This condition, together with the assumption $f(u) \sim u|u|^q$, show that the natural phase space for the energy solutions of (1.6) is the following one:

$$\mathcal{E} = \mathcal{E}(q) := \big[H^1_0(\Omega) \cap L^{q+2}(\Omega)\big] \times L^2(\Omega), \qquad \xi_u(t) := \big(u(t), \partial_t u(t)\big) \in \mathcal{E}.$$

Note that, in the "subcritical" case $q \leqslant 4$, the energy phase space $\mathcal{E}(q)$ is independent of q (due to the embedding $H^1_0(\Omega) \subset L^6(\Omega)$) and coincides with the energy space for the linear problem:

$$\mathcal{E}(q) \equiv H^1_0(\Omega) \cap L^2(\Omega), \quad q \leqslant 4, \tag{1.8}$$

but in the "supercritical" case $q > 4$, the energy phase space $\mathcal{E}(q)$ essentially depends on the growth exponent q.

The existence of such solutions can be easily shown, e.g., by using the Galerkin method (without any restriction on the exponent q). But the standard approach gives the uniqueness only when $q \leqslant 4$ and asymptotic smoothness as well as the existence and finite-dimensionality of the associated global attractor only for $q < 4$ (see [14,32,34,36,37] and references therein). The case $q = 4$ seemed more delicate and has been completely understood only recently (see [10,50,56]). In particular, the existence of a compact global attractor for the *energy* solutions of problem (1.6) in any space dimensions $\Omega \subset \mathbb{R}^n$ and the growth rate $q = \frac{4}{n-2}$ (which corresponds to $q = 4$ for $n = 3$) has been shown in [10] based on the so-called Alekseev non-linear decomposition formula (analogously to [3]). This result is established under slightly weaker assumptions on the non-linear term f than (1.3), namely, the growth restriction of the form

$$\big|f(u) - f(v)\big| \leqslant |u - v|\big(1 + |u|^q + |v|^q\big) \tag{1.9}$$

and the dissipativity assumption of the form

$$\limsup_{|u| \to \infty} \frac{f(u)}{u} \geqslant 0. \tag{1.10}$$

The additional regularity of the constructed global attractor has been established (for $n = 3$) in [50] under the additional quasi-monotonicity assumption

$$f'(u) \geqslant -K$$

exploiting the partial smoothing property (for the $\partial_t u$-component) for Eqs. (1.6) and (1.7). This quasi-monotonicity assumption has been finally removed (even in a more general unbounded domain setting) in [50] using the combination of the above smoothing property with the technique of [62].

The closest to our study are the results and methods of [50] where the regularity of the global attractor in the energy phase space for three-dimensional problem (1.6) with non-linear term growing as $u|u|^4$ is established, see also [49] and [56].

Thus, to the best of our knowledge, the growth exponent $q = 4$ has been usually considered as a critical one for establishing the uniqueness and asymptotic regularity of *energy* solutions for (1.6) and, in particular, the well-posedness of energy solutions for problem (1.6) for the model non-linearities like

$$f(u) = u|u|^q - Cu, \quad q > 4,$$

has been unknown. In the present paper, we remove this restriction and verify the above well-posedness and asymptotic regularity of energy solutions without any restrictions on the exponent q (see Proposition 5.8 for more details). It worth to emphasize here once more that the "linear" phase space (1.8) is *no more relevant* for considering the energy solutions if the growth rate $q > 4$ (since it does not guarantee the finiteness of the energy) and is naturally replaced by the energy space $\mathcal{E}(q) := [H^1_0 \cap L^{q+1}] \times L^2$ in a complete agreement with the particular form of the energy functional. Note also, that the restrictions on f (see (1.3)) posed here are a slightly *stronger* than the assumptions (1.9) and (1.10) for the case $q \leqslant 4$ (and we do not know whether or not the result holds under the assumptions (1.9) and (1.10)).

The quasi-linear case $\phi \neq 0$ is much more complicated. However, in the case of one spatial dimension, more or less complete theory of this equation can be found in the literature (see [14,18,33,40] and references therein). The study of the analytic properties and long-time behavior of solutions of such quasi-linear problems was initiated by the paper of Greenberg, MacCamy and Mizel [33], where the authors considered initial–boundary value problem for the equation

$$\rho_0 u_{tt} = \phi'(u_x)u_{xx} + \lambda u_{xtx}.$$

It is shown that if $\lambda > 0$, and the initial functions are sufficiently smooth, then a unique, smooth solution exists, which moreover goes to zero as t goes to infinity.

The existence of a global attractor for the one-dimensional problem

$$u_{tt} - \alpha u_{xxt} - \partial_x \phi(u_x) + f(u) = g(x), \quad x \in (0,1),\ t > 0,$$

in the phase space $[H^2(\Omega) \cap H^1_0(\Omega)] \times H^1_0(\Omega)$ is established in [14] (under natural conditions on non-linear terms, see also [8]) and the existence of an exponential attractor for that problem in the space $[H^3(\Omega) \cap H^1_0(\Omega)] \times H^1_0(\Omega)$ can be found in [18]. However, even in that relatively simple 1D case, the well-posedness of the problem in the class of *energy solutions* was problematic (if ϕ'' is not globally bounded).

The multi-dimensional case is essentially less understood. Some exception is only the case where the non-linearity ϕ has globally bounded second derivative $\|\phi''(\eta)\| \leqslant C$ (and the non-linearity f satisfies the standard growth restrictions). The typical example of such non-linearity is

$$\phi(\eta) := \sqrt{1 + |\eta|^2}. \tag{1.11}$$

At that case the global existence and uniqueness is straightforward for any space dimension even in the class of energy solutions. Moreover, rewriting Eq. (1.1) in the form of

$$\partial_t u(t) - \gamma \Delta_x u(t) = \partial_t u(0) - \gamma \Delta_x u(0) + \int_0^t \big(\nabla_x \cdot \phi'\big(\nabla_x u(s)\big) + \Delta_x u(s) - f\big(u(s)\big) + g\big)\, ds \tag{1.12}$$

and treating it as a perturbation of a heat equation, one obtains the control of a $W^{2,s}$-norm of the solution u for some $s > n$ which is sufficient to verify the global existence of classical (and even more regular) solutions, see [17,35,39,53,59] and references therein. Note however, that the trick with the integro-differential equation (1.12) works *only* for globally bounded ϕ'' and gives the estimates which grow exponentially (or even doubly exponentially) in time which is not very helpful for the attractor theory.

In contrast to that, the well-posedness of problem (1.1) for the case of *growing* non-linearities ϕ (satisfying (1.2) with $p>1$) has been previously known only for the two-dimensional case. For instance, the case $\Omega=\mathbb{R}^2$ with the non-linearity

$$\phi(\eta):=\Phi_1(\eta_1)+\Phi_2(\eta_2),$$

where the functions $\phi_i:=\Phi_i'$ have at most cubic growth is considered in [51], the case of spherically symmetric non-linearity

$$\phi(\eta)=\tilde{\phi}\big(|\eta|^2\big)$$

with an arbitrary polynomial growth (in a bounded domain $\Omega\subset\mathbb{R}^2$) is studied in [25] and the case of spherically symmetric solutions in any space dimension is investigated in [23,24]. But again, the proof of uniqueness used there requires the C^1-regularity of solutions whose existence is based on the analogue of the integro-differential equation (1.12) and, by this reason, the associated a priori estimates are divergent in time.

It is also worth to mention here the theory of so-called small solutions (which is usual for quasi-linear hyperbolic equations). In particular, Kawashima and Shibata [38] proved global existence and stability of *small* smooth solutions of the initial–boundary value problem for a quasi-linear system of strongly-damped wave equations of the form

$$A_0(U)u_{tt}+\sum_{j=1}^n A_j(U)u_{x_jt}-\sum_{ij=1}^n A_{ij}(U)u_{x_ix_j}=\sum_{ij=1}^n B_{ij}(U)u_{x_ix_jt}$$

with $U=(\nabla u,u_t,\nabla u_t)$ in any space dimension ($A_0(U)$, $A_j(U)$ and $A_{ij}(U)$ are assumed to satisfy some natural assumptions near $U=0$; see also [20]), the analogous results for less regular solutions of the equation

$$u_{tt}-\nabla_x\cdot\left(\frac{\nabla_x u}{\sqrt{1+|\nabla_x u|^2}}\right)-\Delta u_t-|u|^\alpha u=0, \tag{1.13}$$

where α is a given positive number are established in [35].

Thus, to the best of our knowledge, the methods developed in the previous papers were *insufficient* for the attractor theory of the quasi-linear equation (1.1) in the multi-dimensional case and, by that reason, this problem has not been considered in the literature. However, a lot of related results for the simplified versions of Eq. (1.1) can be found: we would like to mention here the papers [6,47,48] devoted to global behavior of solutions of the so-called Kirchhoff equation with strongly damping term (an equation of the form (1.1) with $\phi(\nabla_x u):=\Phi(\|\nabla_x u\|^2_{L^2})$) and the papers [7,52] devoted to the global behavior of solutions for the following operator differential equation in a Hilbert space $\mathcal{H}$:

$$u_{tt}+A_1u+A_2u_t+N^*g(Nu)=h(t), \tag{1.14}$$

where A_1, A_2 and N are given linear operators and $g(\cdot)$ is a non-linear operator. Note that, although our equation (1.1) can be considered as a particular case of (1.14) with $A_1=A_2:=-\Delta_x$ and $N:=\nabla_x$, that choice of operators does not satisfy the assumptions of the aforementioned papers (in particular, the operator $A_2^{-1/2}N$ is not compact).

In the present paper, we give a comprehensive study of Eq. (1.1) in a 3D bounded domain with smooth boundary including the well-posedness of weak energy solutions, asymptotic compactness, existence and finite-dimensionality of global attractors, further regularity of solutions, etc. In particular, we present here a new method of verifying the uniqueness of energy solutions which allows us not only to treat the general quasi-linear equation (1.1), but also to improve essentially the theory in the semi-linear case $\phi\equiv 0$ (by removing the long-standing growth restriction for the non-linearity f).

In order to avoid the additional technicalities, we pay the main attention to the most complicated case of growing non-linearities ϕ (with $p \geqslant 1$) and, by this reason, we have written the additional term $\Delta_x u$ in Eq. (1.1) which automatically excludes the case of degenerate equations as well as the case of the non-linearity (1.11) and Eq. (1.13) (the most part of our results can be extended to the case of Eq. (1.13), but the additional accuracy related with the "non-coercivity" of the non-linearity (1.11) is required). Moreover, for simplicity, we restrict ourselves to consider the most physical three-dimensional case ($\Omega \subset \mathbb{R}^3$) although most of our results can be naturally extended to the case of higher (arbitrary) space dimension. The only difference here is that in higher dimensions ($n > 4$) the growth of the non-linearity f should be also restricted by the second limit exponent q_0 ($p < p_0$ and $q < q_0$) and, of course, the values of the limit exponents $p_0 = p_0(n)$ and $q_0 = q_0(n)$ depend on the space dimension.

The paper is organized as follows.

The well-posedness and dissipativity of weak energy solutions are studied in Section 2. We recall that the energy functional for Eq. (1.1) reads

$$E(u, \partial_t u) := \frac{1}{2}\|\partial_t u\|_{L^2}^2 + \frac{1}{2}\|\nabla_x u\|_{L^2}^2 + \big(\phi(\nabla_x u), 1\big) + \big(F(u), 1\big) - (g, u), \tag{1.15}$$

where $F(v) := \int_0^v f(z)\,dz$ and $(\cdot,\cdot)$ stands for the inner product in $L^2(\Omega)$ and, therefore, the natural choice of the energy phase space is the following one:

$$\mathcal{E} = \mathcal{E}(p, q) := \big[W_0^{1,p+1}(\Omega) \cap L^{q+2}(\Omega)\big] \times L^2(\Omega). \tag{1.16}$$

Here and below, we denote by $W^{s,p}(\Omega)$, $s \geqslant 0$, $1 < p < \infty$, the Sobolev space of distributions whose derivative up to order s belong to $L^p(\Omega)$, as usual $W_0^{s,p}(\Omega)$ means the closure of $C_0^\infty(\Omega)$ in the metric of $W^{s,p}(\Omega)$ and the space $W^{-s,p}(\Omega)$ is defined as a dual space to $W_0^{s,q}(\Omega)$, where $1/p + 1/q = 1$. In addition, we will often use the notation $H^s(\Omega)$ for the space $W^{s,2}(\Omega)$ and the notation $H^s = H^s = D((-\Delta_x)^{s/2})$ for the scale of Hilbert spaces generated by the Laplacian with the Dirichlet boundary conditions.

Note that the energy phase space now depends on *two* growth exponents p and q and the particular case $p = 1$ corresponds to the energy space $\mathcal{E}(q) = \mathcal{E}(1, q)$ associated with the semi-linear problem. In addition, analogously to the semi-linear case, the space $\mathcal{E}(p, q)$ is independent of q if $\frac{1}{q+2} \geqslant \frac{1}{p+1} - \frac{1}{n}$.

To be more precise, we define energy solutions of problem (1.1) as follows.

Definition 1.1. A weak energy solution of the problem (1.1) is a function $u = u(t, x)$ such that

$$(u, \partial_t u) \in L^\infty\big([0, T], \mathcal{E}(p, q)\big), \qquad \partial_t u \in L^2\big([0, T], W^{1,2}(\Omega)\big) \tag{1.17}$$

and which solves (1.1) in the sense of distributions.

We note that, as usual, the trajectory $t \to \xi_u(t) := (u(t), \partial_t u(t))$ is weakly continuous with respect to t as the $\mathcal{E}(p, q)$-valued function. By this reason, the initial data at $t = 0$ is well defined.

The main result of this section is the following theorem.

Theorem 1.2. *Let the non-linearities ϕ and f satisfy assumptions* (1.2) *and* (1.3)*, respectively. Then, for any $\xi_u(0) \in \mathcal{E}(p, q)$, problem* (1.1) *has a unique weak energy solution $\xi_u(t)$ and this solution satisfies the dissipative estimate*:

$$\big\|\xi_u(t)\big\|_{\mathcal{E}}^2 + \int_t^{t+1} \big\|\nabla_x \partial_t u(s)\big\|_{L^2}^2\,ds \leqslant Q\big(\big\|\xi_u(0)\big\|_{\mathcal{E}}^2\big)e^{-\alpha t} + Q\big(\|g\|_{L^2}\big) \tag{1.18}$$

for some positive constant α and monotone function Q independent of u and t.

In a slight abuse of notations, we denote by $\|\xi_u(t)\|_{\mathcal{E}}$ the following energy "norm" in the energy space $\mathcal{E} = \mathcal{E}(p,q)$:

$$\|\xi_u(t)\|^2_{\mathcal{E}} := \|\partial_t u(t)\|^2_{L^2} + \|\nabla_x u(t)\|^{p+1}_{L^{p+1}} + \|u(t)\|^{q+2}_{L^{q+2}}.$$

Moreover, a number of important additional results, including the Lipschitz continuity with respect to the initial data (in a slightly weaker norm) and a partial smoothing property for the $\partial_t u$-component of the solution $\xi_u(t)$ are obtained there.

The existence of a global attractor $\mathcal{A}$ for the semigroup $S(t)$ associated with weak energy solutions of (1.1) is verified in Section 3 using the so-called method of l-trajectories. The attractor $\mathcal{A}$ attracts the images of bounded subsets in $\mathcal{E}(p,q)$ only in the topology of slightly weaker space $\tilde{\mathcal{E}} := W^{1,2}_0(\Omega) \times L^2(\Omega)$ and has finite fractal dimension in $\tilde{\mathcal{E}}$. This drawback is partially corrected below (in Section 5) where the attraction property in the initial phase space is verified for the semi-linear case $\phi \equiv 0$ (and arbitrarily growing non-linearity f).

The so-called *strong* solutions of problem (1.1) are considered in Section 4. By definition, that solutions belong to the space

$$\mathcal{E}_1 := \{u \in W^{2,2}(\Omega) \cap W^{1,2}_0(\Omega),\ \partial_t u \in W^{1,2}_0(\Omega),\ \nabla_x(\phi'(\nabla_x u)) \in W^{-1,2}(\Omega)\} \tag{1.19}$$

for every $t \geqslant 0$ (see Remark 4.5 below for more information about the structure of this space) and the "norm" in that space is naturally defined by

$$\|\xi_u(t)\|^2_{\mathcal{E}_1} := \|\partial_t u(t)\|^2_{H^1} + \|u(t)\|^2_{W^{2,2}} + \|\nabla_x(\phi'(\nabla_x u(t)))\|^2_{W^{-1,2}}.$$

The main result of this section is the following theorem.

Theorem 1.3. *Let the assumptions of Theorem* 1.2 *hold and let, in addition,* $\xi_u(0) \in \mathcal{E}_1$. *Then the associated weak energy solution of* (1.1) *is, in fact, a strong solution and the following estimate holds*:

$$\|\xi_u(t)\|_{\mathcal{E}_1} + \int_t^{t+1} \|\partial_t^2 u(s)\|^2_{L^2}\, ds \leqslant Q(\|\xi_u(0)\|_{\mathcal{E}_1})e^{-\alpha t} + Q(\|g\|_{L^2}) \tag{1.20}$$

for some positive constant α *and monotone function* Q *independent of* t.

The proof of that theorem is based on a combination of our new "uniqueness technique" (which is now used for obtaining the proper estimates for $\partial_t u$) with the technique of [24] extended to the spherically non-symmetric case (which allows us to estimate the $W^{2,2}$-norm of the solution). As usual, the $W^{2,2}$-estimate is based on the multiplication of Eq. (1.1) by $\Delta_x u$ (and works without any growth restrictions on p), however, in contrast to [24], we would prefer to use some kind of non-linear localization technique rather than direct estimates of the boundary terms arising after the integration by parts in the term $(\nabla_x \cdot \phi'(\nabla_x u), \Delta_x u)$, see Lemma 4.1 for details.

In addition, the well-posedness and dissipativity of problem (1.1) for the *critical* growth exponent $p = 4$ for the non-linearity ϕ in the class of *strong* solutions are verified in that section.

However, we have to emphasize that the strong solutions are a priori not regular enough in order to prevent possible singularities of the gradient $\nabla_x u$. So, the global existence and dissipativity of *classical* (and more regular) solutions remains an open problem here.

The asymptotic regularity of weak energy solutions and the existence of an *exponential* attractor of the semigroup $S(t)$ associated with weak energy solutions are verified in Section 5. In particular, using the extension of the technique, developed in [50] to the quasi-linear case, we establish that the proper ball $\mathbb{B}$ in the space $\mathcal{E}_1$ is an exponentially attracting set for the semigroup $S(t)$, i.e., that, for any bounded set $B \subset \mathbb{B}$,

$$\operatorname{dist}_{\tilde{\mathcal{E}}}\big(S(t)B, \mathbb{B}\big) \leqslant Q\left(\|B\|_{\mathcal{E}}\right)e^{-\alpha t}, \tag{1.21}$$

where the positive constant α and monotone function Q are independent of B.

This result immediately implies that the above mentioned global attractor $\mathcal{A}$ consists of *strong* solutions and together with the results of Section 3 gives the existence of an exponential attractor $\mathcal{M}$ for the solution semigroup.

In addition, in the particular semi-linear case $\phi \equiv 0$, we prove the analogue of estimate (1.21) for the initial phase space $\mathcal{E}(q)$ which gives the attraction property in the topology of the initial phase space as well. Thus, we have constructed the complete theory for the semi-linear case with arbitrary polynomial growth rate of the non-linearity f.

Finally, in the concluding Section 6, we discuss the applications of our technique to the related, but more simple equations, including the so-called Kirchhoff equation, membrane equation, and the semi-linear wave equation with structural damping.

It is also worth to note that, although only the case of a bounded domain Ω is considered in the paper, the result can be extended to the case of unbounded domains Ω. We will return to that question somewhere else.

2. Energy solutions: Existence, uniqueness and dissipativity

In this section, we start to study weak energy solutions of problem (1.1). The following standard theorem gives the solvability and dissipativity of the problem (1.1).

Theorem 2.1. *Let the non-linearities ϕ and f satisfy assumptions* (1.2) *and* (1.3), $g \in L^2(\Omega)$ *and* $\xi_u(0) \in \mathcal{E} = \mathcal{E}(p,q)$. *Then, problem* (1.1) *possesses at least one weak energy solution $\xi_u(t)$ which satisfies the following dissipative estimate*:

$$\left\|\xi_u(t)\right\|^2_{\mathcal{E}} + \int_t^{t+1} \left\|\partial_t \nabla_x u(s)\right\|^2_{L^2} ds \leqslant Q\left(\left\|\xi_u(0)\right\|^2_{\mathcal{E}}\right)e^{-\beta t} + Q\left(\|g\|_{L^2}\right), \tag{2.1}$$

where β is a positive constant and Q is a monotone function both independent of the initial data $\xi_u(0)$.

Proof. Since the result of the theorem is more or less standard, we restrict ourselves by only formal derivation of estimate (2.1) which can be easily justified using, e.g., the Galerkin approximation method and the monotonicity arguments for passing to the limit in the quasi-linear term, see [5] for details.

Indeed, multiplying Eq. (1.1) by $\partial_t u$ and integrating over $x \in \Omega$, we arrive at

$$\frac{d}{dt}\left(\frac{1}{2}\|\partial_t u\|^2_{L^2} + \big(\phi(\nabla_x u), 1\big) + \frac{1}{2}\|\nabla_x u\|^2_{L^2} + \big(F(u), 1\big) - (g, u)\right) + \gamma\|\nabla_x \partial_t u\|^2_{L^2} = 0. \tag{2.2}$$

Here $F(u) := \int_0^u f(v)\,dv$ and $(\cdot,\cdot)$ stands for the usual inner product in $L^2(\Omega)$.

Multiplying Eq. (1.1) by αu, where α is a small positive number which will be fixed below, we obtain

$$\frac{d}{dt}\left(\alpha\frac{\gamma}{2}\|\nabla_x u\|^2_{L^2} + \alpha(\partial_t u, u)\right) - \alpha\|\partial_t u\|^2 + \alpha\|\nabla_x u\|^2_{L^2} + \alpha\big(\phi'(\nabla_x u), \nabla_x u\big) + \alpha\big(f(u), u\big)$$
$$= \alpha(g, u). \tag{2.3}$$

Taking a sum of that two equations and denoting

$$E\big(\xi_u(t)\big):=\frac{1}{2}\|\partial_t u\|^2_{L^2}+\big(\phi(\nabla_x u),1\big)+\frac{1}{2}\|\nabla_x u\|^2_{L^2}+\big(F(u),1\big)-(g,u)+\alpha\frac{\gamma}{2}\|\nabla_x u\|^2_{L^2}+\alpha(\partial_t u,u),$$

we deduce that

$$\frac{d}{dt}E(\xi_u)+\gamma\|\nabla_x\partial_t u\|^2_{L^2}-\alpha\|\partial_t u\|^2_{L^2}+\alpha\|\nabla_x u\|^2_{L^2}+\alpha\big(\phi'(\nabla_x u),\nabla_x u\big)+\alpha\big(f(u),u\big)=\alpha(g,u).$$

We fix α be small enough that

$$\begin{aligned}&\beta\big(\|\partial_t u\|^2_{L^2}+\|\nabla_x u\|^{p+1}_{L^{p+1}}+\|u\|^{q+2}_{L^{q+2}}\big)-C\big(1+\|g\|^2_{L^2}\big)\\&\quad\leqslant E(\xi_u)\leqslant C\big(1+\|g\|^2_{L^2}\big)+\|\partial_t u\|^2_{L^2}+\|\nabla_x u\|^{p+1}_{L^{p+1}}+\|u\|^{q+2}_{L^{q+2}}\end{aligned}\tag{2.4}$$

and that $\|\partial_t\nabla_x u\|^2_{L^2}\geqslant 2\alpha\|\partial_t u\|^2_{L^2}$ (it is possible to do due to our assumptions on ϕ and f, see (1.2) and (1.3)). Then, the last inequality gives

$$\frac{d}{dt}E\big(\xi_u(t)\big)+\delta E\big(\xi_u(t)\big)+\delta\|\partial_t\nabla_x u\|^2_{L^2}\leqslant C\big(1+\|g\|^2_{L^2}\big)$$

for some positive constants δ and C. Applying the Gronwall inequality to this estimate and using (2.4), we arrive at the desired estimate (2.1) and finish the proof of the theorem. $\square$

Note that the growth restriction $p<5$ is nowhere used in the proof of Theorem 2.1 and, therefore, that result holds for arbitrary growth p. In contrast to that, the next uniqueness result essentially uses this growth restriction.

Theorem 2.2. *Let the assumptions of Theorem* 2.1 *hold. Then, the energy solution of problem* (1.1) *is unique. Moreover, for every two energy solutions $u_1(t)$ and $u_2(t)$* (*with different initial data*)*, the following Lipschitz continuity in a weaker space holds*:

$$\big\|\partial_t v(t)\big\|^2_{H^{-1}}+\big\|v(t)\big\|^2_{H^1}\leqslant Ce^{Kt}\big(\big\|\partial_t v(0)\big\|^2_{H^{-1}}+\big\|v(0)\big\|^2_{H^1}\big),\tag{2.5}$$

where $v(t):=u_1(t)-u_2(t)$ and the constants C and K depend only on the $\mathcal{E}(p,q)$-norms of the initial data.

Proof. Indeed, the function $v(t)$ solves

$$\begin{cases}\partial_t^2 v-\gamma\partial_t\Delta_x v-\Delta_x v+\big[f(u_1)-f(u_2)\big]=\nabla_x\big(\phi'(\nabla_x u_1)-\phi'(\nabla_x u_2)\big),\\ v|_{\partial\Omega}=0,\quad v(0)=u_1(0)-u_2(0),\quad \partial_t v(0)=\partial_t u_1(0)-\partial_t u_2(0).\end{cases}\tag{2.6}$$

In order to estimate the non-linear terms in that equation, we need the following lemma.

Lemma 2.3. *(See for instance [19] or [60].) If the function ϕ satisfies conditions* (1.2) *then there exists a constant $\delta>0$ such that*

$$\big[\phi'(\eta_1)-\phi'(\eta_2)\big]\cdot[\eta_1-\eta_2]\geqslant\delta\big(|\eta_1|+|\eta_2|\big)^{p-1}|\eta_1-\eta_2|^2,\quad\forall\eta_1,\eta_2\in\mathbb{R}^3.\tag{2.7}$$

Thus, due to our assumptions on f and Lemma 2.3,

$$\big[f(u_1)-f(u_2)\big].(u_1-u_2)\geqslant -C|v|^2+d_0\big(|u_1|^q+|u_2|^q\big)|v|^2$$

for some positive d_0. Analogously, for the function ϕ', we have

$$\big(\phi'(\nabla_x u_1)-\phi'(\nabla_x u_2),\nabla_x v\big)\geqslant d_0\big(\big(|\nabla_x u_1|+|\nabla_x u_2|\big)^{p-1},|\nabla_x v|^2\big).$$

Multiplying Eq. (2.6) by v, integrating over $x\in\Omega$ (it is not difficult to check that all of the integrals have sense; as usual, the regularity of the energy solution is sufficient for multiplication of the equation by u, but not for $\partial_t u$) and using the above estimates for the functions f and ϕ, we will have

$$\begin{aligned}&\partial_t\left[\frac{\gamma}{2}\|\nabla_x v\|^2_{L^2}+(v,\partial_t v)\right]+d_0\big(\big(|\nabla_x u_1|+|\nabla_x u_2|\big)^{p-1}+1,|\nabla_x v|^2\big)+d_0\big(\big(|u_1|+|u_2|\big)^q,|v|^2\big)\\&\quad\leqslant C\|v\|^2_{L^2}+\|\partial_t v\|^2_{L^2}\end{aligned}\tag{2.8}$$

for some positive d_0.

In order to control the term with $\partial_t v$ in the right-hand side of (2.8), we multiply Eq. (2.6) by $(-\Delta_x)^{-1}\partial_t v$ and integrate over $x\in\Omega$ (again, all of the integrals will have sense). Then, we have

$$\begin{aligned}&\partial_t\big(\|\partial_t v\|^2_{H^{-1}}+\|v\|^2_{L^2}\big)+2\gamma\|\partial_t v\|^2_{L^2}\\&\quad\leqslant 2\big(\big|f(u_1)-f(u_2)\big|,\big|(-\Delta_x)^{-1}\partial_t v\big|\big)+2\big(\big|\phi'(\nabla_x u_1)-\phi'(\nabla_x u_2)\big|,\big|\nabla_x(-\Delta_x)^{-1}\partial_t v\big|\big).\end{aligned}\tag{2.9}$$

Using the growth assumptions on f and the embedding $W^{2,2-s}(\Omega)\subset C(\Omega)$, for $s\in[0,\frac{1}{2})$, the first term in the right-hand side of (2.9) can be estimated as follows:

$$\begin{aligned}&\big(\big|f(u_1)-f(u_2)\big|,\big|(-\Delta_x)^{-1}\partial_t v\big|\big)\\&\quad\leqslant\varepsilon\big\|f(u_1)-f(u_2)\big\|^2_{L^1}+C_\varepsilon\|\partial_t v\|^2_{H^{-s}}\\&\quad\leqslant\varepsilon\big(\big(|u_1|+|u_2|\big)^q,|v|\big)^2+\varepsilon\|\partial_t v\|^2_{L^2}+C_\varepsilon\|\partial_t v\|^2_{H^{-1}}\\&\quad\leqslant\varepsilon\big(|u_1|^q+|u_2|^q,1\big)\big(|u_1|^q+|u_2|^q,|v|^2\big)+\varepsilon\|\partial_t v\|^2_{L^2}+C_\varepsilon\|\partial_t v\|^2_{H^{-1}}\\&\quad\leqslant\varepsilon\big(\|\xi_{u_1}\|_{\mathcal{E}}+\|\xi_{u_2}\|_{\mathcal{E}}\big)^q\big(\big(|u_1|+|u_2|\big)^q,|v|^2\big)+\varepsilon\|\partial_t v\|^2_{L^2}+C_\varepsilon\|\partial_t v\|^2_{H^{-1}},\end{aligned}\tag{2.10}$$

where $\varepsilon>0$ is arbitrary.

Analogously, using the growth restriction $p+1<6$, together with the Hölder inequality and the interpolation

$$\big\|\nabla_x(-\Delta_x)^{-1}\partial_t v\big\|^2_{L^{p+1}}\leqslant\varepsilon\big\|\nabla_x(-\Delta_x)^{-1}\partial_t v\big\|^2_{H^1}+C_\varepsilon\big\|\nabla_x(-\Delta_x)^{-1}\partial_t v\big\|^2_{L^2}=\varepsilon\|\partial_t v\|^2_{L^2}+C_\varepsilon\|\partial_t v\|^2_{H^{-1}},$$

one can estimate the second term in the right-hand side of (2.9) as follows:

$$\begin{aligned}&\big(\big|\phi'(\nabla_x u_1)-\phi'(\nabla_x u_2)\big|,\big|\nabla_x(-\Delta_x)^{-1}\partial_t v\big|\big)\\&\quad\leqslant\varepsilon\big\|\phi(\nabla_x u_1)-\phi(\nabla_x u_2)\big\|^2_{L^{(p+1)/p}}+\varepsilon\big\|\nabla_x(-\Delta_x)^{-1}\partial_t v\big\|^2_{L^{p+1}}\\&\quad\leqslant\varepsilon\big(\big(|\nabla_x u_1|+|\nabla_x u_2|\big)^{(p^2-1)/p},|\nabla_x v|^{(p+1)/p}\big)^{2p/(p+1)}+\varepsilon\|\partial_t v\|^2_{L^2}+C_\varepsilon\|\partial_t v\|^2_{H^{-1}}\\&\quad\leqslant\varepsilon\big(\big(|\nabla_x u_1|+|\nabla_x u_2|\big)^{p+1},1\big)^{(p-1)/(p+1)}\big(\big(|\nabla_x u_1|+|\nabla_x u_2|\big)^{p-1},|\nabla_x v|^2\big)\end{aligned}$$

$$+\varepsilon\|\partial_t v\|_{L^2}^2 + C_\varepsilon\|\partial_t v\|_{H^{-1}}^2 \varepsilon\big(\|\xi_{u_1}\|_{\mathcal{E}} + \|\xi_{u_2}\|_{\mathcal{E}}\big)^{p-1}$$
$$\times\big(\big(|\nabla_x u_1| + |\nabla_x u_2|\big)^{p-1}, |\nabla_x v|^2\big) + \varepsilon\|\partial_t v\|_{L^2}^2 + C_\varepsilon\|\partial_t v\|_{H^{-1}}^2, \tag{2.11}$$

where $\varepsilon > 0$ is again arbitrary.

Inserting estimates (2.10) and (2.11) into the right-hand side of (2.9), taking a sum with estimate (2.8) multiplied by $\frac{2}{\gamma}$, using the energy estimate (2.1) for estimating the energy norms of ξ_{u_i} and fixing $\varepsilon > 0$ small enough, we finally deduce that

$$\partial_t\left[\frac{\gamma}{2}\|\nabla_x v\|_{L^2}^2 + (\partial_t v, v) + \frac{2}{\gamma}\big(\|\partial_t v\|_{H^{-1}}^2 + \|v\|_{L^2}^2\big)\right] + 2\beta\big(\|\partial_t v\|_{L^2}^2 + \|\nabla_x v\|_{L^2}^2\big)$$
$$\leqslant C\big(\|v\|_{L^2}^2 + \|\partial_t v\|_{H^{-1}}^2\big), \tag{2.12}$$

where the constant C depends on the energy norms of $\xi_{u_1}(0)$ and $\xi_{u_2}(0)$. Moreover, obviously, the function

$$E_{-1}(v) := \frac{\gamma}{2}\|\nabla_x v\|_{L^2}^2 + (\partial_t v, v) + \frac{2}{\gamma}\big(\|\partial_t v\|_{H^{-1}}^2 + \|v\|_{L^2}^2\big) \tag{2.13}$$

satisfies

$$\kappa_1\big(\|\partial_t v\|_{H^{-1}}^2 + \|v\|_{H^1}^2\big) \leqslant E_{-1}(v) \leqslant \kappa_2\big(\|\partial_t v\|_{H^{-1}}^2 + \|v\|_{H^1}^2\big) \tag{2.14}$$

for some positive κ_i. Applying the Gronwall inequality to (2.12), we see that

$$\big\|\partial_t v(t)\big\|_{H^{-1}}^2 + \big\|v(t)\big\|_{H^1}^2 \leqslant Ce^{Kt}\big(\big\|\partial_t v(0)\big\|_{H^{-1}}^2 + \big\|v(0)\big\|_{H^1}^2\big) \tag{2.15}$$

and the uniqueness holds. $\square$

Thus, for every $\xi_u(0) \in \mathcal{E}(p,q)$, there exists a unique weak solution $\xi_u(t)$ of problem (1.1) and, therefore, the solution semigroup

$$S(t) : \mathcal{E}(p,q) \to \mathcal{E}(p,q), \qquad S(t)\xi_u(0) := \xi_u(t), \quad t \geqslant 0, \tag{2.16}$$

is well defined in the energy space $\mathcal{E}(p,q)$. Moreover, as the proved theorem shows, this semigroup is Lipschitz continuous in the weaker space $\mathcal{E}_{-1} := H_0^1(\Omega) \times H^{-1}(\Omega)$:

$$\big\|S(t)\xi_1 - S(t)\xi_2\big\|_{\mathcal{E}_{-1}} \leqslant Ce^{Kt}\|\xi_1 - \xi_2\|_{\mathcal{E}_{-1}}, \tag{2.17}$$

where the constants C and K depend on the energy norms of ξ_1 and ξ_2. We will essentially use below this weak Lipschitz continuity in order to prove the finite-dimensionality of the global attractor.

Our next task is to establish some partial smoothing property of the constructed semigroup $S(t)$. We start with the case where the $\partial_t u$-component of the initial data is more regular.

Proposition 2.4. *Let the assumptions of Theorem* 2.2 *be satisfied and let the initial data $\xi_u(0)$ be such that*

$$\partial_t u(0) \in H_0^1(\Omega), \qquad \partial_t^2 u(0) := \gamma\partial_t\Delta_x u(0) + \Delta_x u(0) + \nabla_x\phi'\big(\nabla_x u(0)\big) - f\big(u(0)\big) + g \in H^{-1}(\Omega). \tag{2.18}$$

Then, the function $v(t) := \partial_t u(t)$ is such that $\xi_v(t) \in \mathcal{E}_{-1}$ for every $t \geqslant 0$ and the following estimate holds:

$$\|\xi_v(t)\|^2_{\mathcal{E}_{-1}} + \int_t^{t+1} \|\partial_t v(s)\|^2_{L^2}\, ds \leqslant Q\big(\|\xi_v(0)\|_{\mathcal{E}_{-1}} + \|\xi_u(0)\|_{\mathcal{E}}\big)e^{-\gamma t} + Q\big(\|g\|_{L^2}\big), \tag{2.19}$$

where the positive constant C and monotone function Q are independent of t and u.

Proof. As in the proof of Theorem 2.1, we give below only formal derivation of the estimate (2.19) which can be justified, e.g., by the Galerkin approximation method.

Indeed, the function $v(t)$ solves

$$\partial_t^2 v - \gamma\partial_t\Delta_x v - \Delta_x v + f'(u)v = \nabla_x\big(\phi''(\nabla_x u)\nabla_x v\big), \qquad \xi_v(0) \in \mathcal{E}_{-1} \tag{2.20}$$

which is of the form (2.6). By this reason, multiplying this equation by the function $v(t) + \frac{2}{\gamma}(-\Delta_x)^{-1}\partial_t v(t)$ and arguing exactly as in the derivation of (2.12), we end up with the following inequality:

$$\begin{aligned} &\partial_t E_{-1}\big(v(t)\big) + \beta E_{-1}\big(v(t)\big) + \beta\big(\|\partial_t v(t)\|^2_{L^2} + \|\nabla_x v(t)\|^2_{L^2}\big) \\ &\quad \leqslant Q\big(\|\xi_u(t)\|_{\mathcal{E}}\big)\big(\|v(t)\|^2_{L^2} + \|\partial_t v(t)\|^2_{H^{-1}}\big), \end{aligned} \tag{2.21}$$

where $\beta > 0$ is a fixed constant, Q is a given monotone function and the functional $E_{-1}(v)$ is given by (2.13). Moreover, using the embedding $L^1 \subset H^{-2}$ and expressing $\partial_t^2 u$ from Eq. (1.1), we have

$$\|\partial_t^2 v(t)\|_{H^{-3}} \leqslant Q\big(\|\xi_u(t)\|_{\mathcal{E}}\big) + \|g\|_{L^2} \tag{2.22}$$

for some monotone function Q. Indeed, taking the inner product of Eq. (2.13) with arbitrary test function $\varphi \in H^3 = D((-\Delta_x)^{3/2})$, we have

$$\big(\partial_t^2 u, \varphi\big) = \gamma(\partial_t v, \Delta_x\varphi) + (v, \Delta_x\varphi) - \big(\phi'(\nabla_x u), \nabla_x\varphi\big) - \big(f(u), \varphi\big) + (g, \varphi).$$

Using now that the L^1-norms of $f(u)$ and $\varphi'(\nabla_x u)$ are controlled by the energy norm and the fact that $H^2 \subset L^\infty$, we deduce estimate (2.22).

This estimate, together with the interpolation inequality

$$\|\partial_t v(t)\|_{H^{-1}} \leqslant C\|\partial_t v(t)\|^{1/3}_{H^{-3}}\|\partial_t v(t)\|^{2/3}_{L^2}, \tag{2.23}$$

allows us to control the right-hand side of (2.21)

$$\partial_t E_{-1}\big(v(t)\big) + \beta E_{-1}\big(v(t)\big) + \frac{1}{2}\beta\big(\|\partial_t v\|^2_{L^2} + \|\nabla_x v\|^2_{L^2}\big) \leqslant Q\big(\|\xi_u(t)\|_{\mathcal{E}}\big) \tag{2.24}$$

for some new monotone function Q. Applying the Gronwall inequality to this relation and using (2.1) to control the right-hand side, we derive the required estimate (2.19) and finish the proof of the proposition. □

Finally, the next proposition shows that the $\partial_t u$-component of the energy solution becomes more regular for $t > 0$ (in a complete agreement with the semi-linear case, see [50]).

Proposition 2.5. *Let the assumptions of Theorem 2.2 hold and let $u(t)$ be a weak energy solution of Eq. (1.1). Denote $v(t) := \partial_t u(t)$. Then, $\xi_v(t) \in \mathcal{E}_{-1}$ for any $t > 0$ and the following estimate holds:*

$$\left\|\partial_t u(t)\right\|^2_{H^1}+\left\|\partial_t^2 u(t)\right\|^2_{H^{-1}}+\int_t^{t+1}\left\|\partial_t^2 u(s)\right\|^2_{L^2}\,ds\leqslant\frac{1+t^N}{t^N}Q\left(\left\|\xi_u(0)\right\|_{\mathcal{E}}\right)e^{-\beta t}+Q\left(\|g\|_{L^2}\right),\tag{2.25}$$

where $N>0$, $\beta>0$ are some constants and Q is a monotone function which are independent of t and u.

Proof. Indeed, due to estimate (2.19), we only need to verify (2.25) for $t\in(0,1]$. Let us multiply estimate (2.24) by t^3. Then, after the evident transformations, we get

$$\begin{aligned}&\partial_t\left(t^3\left\|\xi_v(t)\right\|^2_{\mathcal{E}_{-1}}\right)+\beta\left(t^3\left\|v(t)\right\|^2_{H^1}+t^3\left\|\partial_t v(t)\right\|^2_{L^2}\right)\\&\quad\leqslant Q\left(\left\|\xi_u(t)\right\|_{\mathcal{E}}\right)\left(1+\left\|\partial_t\nabla_x u(t)\right\|^2_{L^2}\right)+Ct^2\left\|\partial_t v(t)\right\|^2_{H^{-1}},\quad t\in[0,1],\end{aligned}\tag{2.26}$$

for some new monotone function Q and a constant C. We see that all terms in the right-hand side except the last one are under the control due to the energy estimate (2.1). In order to estimate this term, we use the interpolation inequality (2.23) and the Young inequality as follows:

$$t^2\|\partial_t v\|^2_{H^{-1}}\leqslant Ct^2\|\partial_t v\|^{4/3}_{L^2}\|\partial_t v\|^{2/3}_{H^{-3}}\leqslant\varepsilon t^3\|\partial_t v\|^2_{L^2}+C^3\varepsilon^{-2}\left\|\partial_t^2 u\right\|^2_{H^{-3}},$$

where $\varepsilon>0$ is small enough. Inserting this estimate into the right-hand side of (2.26), integrating it over t and using (2.1) and (2.22), we arrive at estimate (2.25) (recall that $t\in(0,1]$) and finish the proof of the proposition. □

3. Weak global attractor and finite-dimensionality

In this section, we will construct the global attractor for the semigroup $S(t)$ generated by weak energy solutions of (1.1) and prove that it has finite fractal dimension. Unfortunately, we are not able to construct the global attractor in a strong topology of $\mathcal{E}(p,q)$, but only in the weaker topology of $\mathcal{E}_{-1}$.

Namely, a set $\mathcal{A}\subset\mathcal{E}(p,q)$ is a weak global attractor of the solution semigroup $S(t):\mathcal{E}(p,q)\to\mathcal{E}(p,q)$ generated by Eq. (1.1) (the so-called $(\mathcal{E}(p,q),\mathcal{E}_{-1})$-attractor in the terminology of Babin and Vishik, see [5]) if

(1) the set $\mathcal{A}$ is bounded in $\mathcal{E}(p,q)$ and compact in $\mathcal{E}_{-1}$;
(2) it is strictly invariant: $S(t)\mathcal{A}=\mathcal{A}$, $t\geqslant 0$;
(3) the set $\mathcal{A}$ attracts the images of bounded in $\mathcal{E}$ sets in the topology of $\mathcal{E}_{-1}$, i.e., for every bounded in $\mathcal{E}(p,q)$ set B and every neighborhood $\mathcal{O}(\mathcal{A})$ of $\mathcal{A}$ in $\mathcal{E}_{-1}$, there is a time $T=T(B,\mathcal{O})$ such that

$$S(t)B\subset\mathcal{O}(\mathcal{A}),\quad t\geqslant T.$$

The following theorem which establishes the existence of such an attractor for semigroup (2.16) is the main result of this section.

Theorem 3.1. *Let the assumptions of Theorem* 2.1 *be satisfied. Then, the solution semigroup $S(t)$ generated by Eq.* (1.1) *in the phase space $\mathcal{E}(p,q)$ possesses a weak global attractor $\mathcal{A}$ in the above sense. The attractor $\mathcal{A}$ consists of all complete bounded trajectories of Eq.* (1.1):

$$\mathcal{A}=\mathcal{K}|_{t=0},\tag{3.1}$$

where $\mathcal{K}\subset L^\infty(\mathbb{R},\mathcal{E})$ is a set of all solutions $\xi_u(t)$ of Eq. (1.1) *which are defined for all $t\in\mathbb{R}$ and bounded: $\|\xi_u(t)\|_{\mathcal{E}}\leqslant C_u$, $t\in\mathbb{R}$.*

Moreover, the attractor $\mathcal{A}$ has finite fractal dimension in $\mathcal{E}_{-1}$:

$$\dim_F(\mathcal{A}, \mathcal{E}_{-1}) \leqslant C < \infty. \tag{3.2}$$

Proof. We first note that, due to the dissipative estimate (2.1), the ball

$$B_R := \{\xi \in \mathcal{E}(p,q),\ \ \|\xi\|_{\mathcal{E}} \leqslant R\}$$

is an absorbing ball for the solution semigroup $\{S(t),\ t \geqslant 0\}$ associated with Eq. (1.1) if R is large enough. Moreover, the same estimate guarantees also that the set

$$\mathcal{B}_R := \Big[\bigcup_{t\geqslant 0} B_R\Big]_{\mathcal{E}_{-1}},$$

where $[\cdot]_V$ denotes the closure in the space V, will be bounded and closed in $\mathcal{E}(p,q)$ absorbing set satisfying the additional semi-invariance property

$$S(t)\mathcal{B}_R \subset \mathcal{B}_R. \tag{3.3}$$

Thus, it is sufficient to construct the global attractor $\mathcal{A}$ for the semigroup $S(t)$ acting on $\mathcal{B}_R$ only. In order to do that, we need to refine estimate (2.12).

Lemma 3.2. *Let $\xi_{u_1}(t)$ and $\xi_{u_2}(t)$ be two trajectories of the semigroup $S(t)$ starting from the absorbing set $\mathcal{B}_u$ and let $v(t) := u_1(t) - u_2(t)$. Then, the following estimate holds*:

$$\begin{aligned}
&\|\xi_v(t)\|^2_{\mathcal{E}_{-1}} + \int_s^t \|\partial_t^2 v(\tau)\|^2_{H^{-3}}\,d\tau \\
&\quad \leqslant Ce^{-\kappa(t-s)}\|\xi_v(s)\|^2_{\mathcal{E}_{-1}} + C\int_s^t \big(\|v(\tau)\|^2_{L^2} + \|\partial_t v(\tau)\|^2_{H^{-2}}\big)\,d\tau,
\end{aligned} \tag{3.4}$$

where the positive constants C and κ are independent of u_1 and u_2 and $t \geqslant s \geqslant 0$.

Proof. Indeed, function $v(t)$ solves Eq. (2.6) and, arguing exactly as in the proof of Theorem 2.2, we verify that estimate (2.12) holds with the constant C independent of the choice of $\xi_{u_1}(0), \xi_{u_2}(0) \in \mathcal{E}(p,q)$. Using now estimate (2.14) together with the interpolation inequality, we infer from (2.12) and (2.8) that

$$\begin{aligned}
&\frac{d}{dt}E_{-1}\big(v(t)\big) + \kappa E_{-1}\big(v(t)\big) + \kappa\big(\phi'\big(\nabla_x u_1(t)\big) - \phi'\big(\nabla_x u_2(t)\big), \nabla_x v(t)\big) \\
&\qquad + \kappa\big(\big|f\big(u_1(t)\big) - f\big(u_2(t)\big)\big|, \big|v(t)\big|\big) \\
&\quad \leqslant C\big(\|v(t)\|^2_{L^2} + \|\partial_t v(t)\|^2_{H^{-2}}\big),
\end{aligned} \tag{3.5}$$

where κ and C are fixed positive constants independent of v. Applying the Gronwall inequality and using (2.13) again, we get

$$\big\|\xi_v(t)\big\|^2_{\mathcal{E}_{-1}}+\int_s^t\big(\phi'\big(\nabla_x u_1(\tau)\big)-\phi'\big(\nabla_x u_2(\tau)\big),\nabla_x v(\tau)\big)+\kappa\big(\big|f\big(u_1(\tau)\big)-f\big(u_2(\tau)\big)\big|,\big|v(\tau)\big|\big)\,d\tau$$

$$\leqslant C_1e^{-\kappa(t-s)}\big\|\xi_v(s)\big\|^2_{\mathcal{E}_{-1}}+C_1\int_s^t\big(\big\|v(\tau)\big\|^2_{L^2}+\big\|\partial_t v(\tau)\big\|^2_{H^{-2}}\big)\,d\tau, \tag{3.6}$$

where κ and C_1 are independent of v and $t\geqslant s\geqslant 0$.

Furthermore, arguing exactly as in (2.10) and (2.11), we see that

$$\begin{aligned}&\big\|\phi'(\nabla_x u_1)-\phi'(\nabla_x u_2)\big\|^2_{L^1}+\big\|f(u_1)-f(u_2)\big\|^2_{L^1}\\&\qquad\leqslant C\big(\phi(u_1)-\phi(u_2),\nabla_x v\big)+C\big(\big|f(u_1)-f(u_2)\big|,|v|\big).\end{aligned} \tag{3.7}$$

Using Eq. (2.6) in order to express the value of $\partial_t^2 v$ together with (3.6), (3.7) and the embedding $L^1\subset H^{-2}$, we deduce the desired estimate (3.4) and finish the proof of the lemma. □

The proved estimate (3.4) is the key technical tool for the so-called method of l-trajectories, see [43]. Indeed, let L be a sufficiently large number which will be fixed below and let

$$\begin{cases}\mathcal{H}_1:=L^2\big([0,L],H^1_0(\Omega)\big)\cap W^{1,2}\big([0,L],H^{-1}(\Omega)\big)\cap W^{2,2}\big([0,L],H^{-3}\big),\\ \mathcal{H}:=L^2\big([0,L],L^2(\Omega)\big)\cap W^{1,2}\big([0,L],H^{-2}\big).\end{cases} \tag{3.8}$$

Then, evidently, the space $\mathcal{H}_1$ is compactly embedded to $\mathcal{H}$. Introduce now the lifting operator

$$\mathbb{T}_L:\mathcal{B}_R\to\mathcal{H}_1,\quad \mathbb{T}_L\xi:=u(\cdot),$$

where u is an L-piece of the trajectory starting from $\xi\in\mathcal{B}_R$. Let also $\mathbb{B}^{tr}:=\mathbb{T}_L\mathcal{B}_R$. Then, the map $\mathbb{T}_L:\mathcal{B}_R\to\mathbb{B}^{tr}$ is one-to-one and we can consider the lift of the solution semigroup $\{S(t),\ t\geqslant 0\}$ to the L-trajectory phase space:

$$\mathbb{S}(t):\mathbb{B}^{tr}\to\mathbb{B}^{tr},\quad \mathbb{S}(t):=\mathbb{T}_L\circ S(t)\circ\mathbb{T}_L^{-1}. \tag{3.9}$$

The idea is now to verify the existence of the global attractor $\mathbb{A}$ of the L-trajectory semigroup $\mathbb{S}(t)$ (more precisely, of the discrete semigroup generated by the map $\mathbb{S}(L)$) in the space $\mathbb{B}^{tr}$ instead of studying the initial semigroup $S(t)$. Indeed, although the map $\mathbb{T}_L^{-1}:\mathbb{B}^{tr}\to\mathcal{B}_h$ is not continuous, it is not difficult to check that the map $T_L^{-1}\circ\mathbb{S}(L)$ is even Lipschitz continuous. To this end, we need to write estimate (2.17) in the form

$$\big\|\xi_v(L)\big\|^2_{\mathcal{E}_{-1}}\leqslant Ce^{K(L-s)}\big\|\xi_v(s)\big\|^2_{\mathcal{E}_{-1}}$$

and integrate it over $s\in[0,L]$. By this reason, it is really sufficient to prove the existence of a finite-dimensional attractor $\mathbb{A}$ for the discrete semigroup generated by the map $\mathbb{S}(L)$ only. The desired finite-dimensional global attractor $\mathcal{A}$ of the initial semigroup can be found then via

$$\mathcal{A}=\mathbb{T}_L^{-1}\circ\mathbb{S}(L)\mathbb{A}$$

and the description (3.1) is a standard corollary of the attractor's existence, see [5].

Finally, for proving the attractor existence for the map $\mathbb{S}(L)$, we note that the basic estimate (3.4) implies the following version of the squeezing property for the map $\mathbb{S}(L)$:

$$\left\|\mathbb{S}(L)\xi_1-\mathbb{S}(L)\xi_2\right\|^2_{\mathcal{H}_1}\leqslant\frac{C_1}{L}\|\xi_1-\xi_2\|^2_{\mathcal{H}_1}+C_L\big(\|\xi_1-\xi_2\|^2_{\mathcal{H}}+\left\|\mathbb{S}(L)\xi_1-\mathbb{S}(L)\xi_2\right\|^2_{\mathcal{H}}\big),\tag{3.10}$$

where $\xi_1,\xi_2\in\mathbb{B}^{tr}$ and the constant C_1 is independent of L. Indeed, integrating estimate (3.4) over $s\in[0,L]$, and fixing $t\in[L,2L]$, we get

$$L\int_L^{2L}\left\|\partial_t^2 v(\tau)\right\|^2_{H^{-3}}d\tau\leqslant Ce^{-\kappa L}\left\|v(\cdot)\right\|^2_{\mathcal{H}_1}+CL\int_0^{2L}\big(\left\|v(\tau)\right\|^2_{L^2}+\left\|\partial_t v(\tau)\right\|^2_{H^{-2}}\big)\,d\tau$$

and

$$L\left\|\xi_v(t)\right\|^2_{\mathcal{E}_{-1}}\leqslant Ce^{-\kappa(t-L)}\left\|v(\cdot)\right\|^2_{\mathcal{H}_1}+CL\int_0^{2L}\big(\left\|v(\tau)\right\|^2_{L^2}+\left\|\partial_t v(\tau)\right\|^2_{H^{-2}}\big)\,d\tau.$$

Integrating the last estimate over $t\in[L,2L]$ and using the previous one, we deduce (3.10).

Fix now L in such a way that $\alpha:=\frac{C_1}{L}<1$. Then, the smoothing property (3.10) together with the obvious Lipschitz continuity of the map $\mathbb{S}(L)$:

$$\left\|\mathbb{S}(L)\xi_1-\mathbb{S}(L)\xi_2\right\|_{\mathcal{H}_1}\leqslant Ce^{KL}\|\xi_1-\xi_2\|_{\mathcal{H}_1},\quad \xi_i\in\mathbb{B}^{tr}$$

(which is also an immediate corollary of (3.4)), imply in a standard way the existence of the global attractor for the map $\mathbb{S}(L)$ and its finite-dimensionality in the space $\mathcal{H}_1$, see [43] (and also [22,61]). Theorem 3.1 is proved. □

Remark 3.3. Note that, usually, the method of L-trajectories gives not only the existence of a global attractor, but also of the *exponential* attractor for the solution semigroup $S(t)$. However, in our case $\mathcal{E}_{-1}$ is not compactly embedded in $\mathcal{E}(p,q)$ and an energy solution is not Hölder continuous in time. This obstacle does not allow to pass from the exponential attractor for the discrete map $S(L)$ (which is factually constructed by the L-trajectory technique) to the desired exponential attractor of the continuous semigroup $S(t)$. We will overcome this obstacle below using the additional smoothness of the global attractor.

Remark 3.4. The space $\mathcal{E}_{-1}:=H^1_0(\Omega)\times H^{-1}(\Omega)$ in Theorem 3.1 can be replaced by the better one $\tilde{\mathcal{E}}:=H^1_0(\Omega)\times L^2(\Omega)$. Indeed, due to Proposition 2.5, we know that $\partial_t u(t)\in H^1_0(\Omega)$ for $t\geqslant 0$. Using this fact and the proper interpolation inequality, we obtain the following Hölder continuity:

$$\left\|S(1)\xi_1-S(1)\xi_2\right\|_{\tilde{\mathcal{E}}}\leqslant C\|\xi_1-\xi_2\|^{1/2}_{\mathcal{E}_{-1}},\quad \xi_1,\xi_2\in\mathcal{B}_R,$$

which immediately gives that result. However, we do not know how to prove that the attractor $\mathcal{A}$ attracts bounded sets in a strong topology of the phase space $\mathcal{E}=\mathcal{E}(p,q)$. For the particular semilinear case ($\phi''(\nabla_x u)=const$ and $f(u)$ with an arbitrary growth rate) this result is established in Section 5.

4. Strong solutions: A priori estimates, existence and dissipativity

This section is devoted to strong solutions of Eq. (1.1). We start with the formal derivation of a dissipative estimate in the space $\mathcal{E}_1$ which will be justified below. To this end, it is convenient to relax slightly our assumptions on the non-linearity ϕ (we are planning to prove the existence of a strong solution by approximating the growing non-linearities by the non-growing ones and, by this reason, we need to allow the non-growing non-linearities). Namely, we replace our assumptions on ϕ by

$$\begin{cases} (1)\ \kappa_1\left|\phi''(v)\right|\cdot|w|^2 \leqslant \phi''(v)w.w \leqslant \kappa_2\left(1+\left|\phi''(v)\right|\right)\cdot|w|^2, \quad w\in\mathbb{R}^3,\\ (2)\ \left|\phi''(v)\right|\cdot|v|^2 \leqslant C\left(1+\phi(v)\right),\\ (3)\ \phi(v)\leqslant C\left(1+\phi'(v).v\right),\\ (4)\ \left|\phi'(v)\right| \leqslant C\left(1+\phi(v)\right)^{1/2}\left|\phi''(v)\right|^{1/2}. \end{cases} \tag{4.1}$$

Since we do not have now any growth restrictions on ϕ, we need to modify the energy norm:

$$\|\xi_u\|^2_{\mathcal{E}_\phi} := \|\partial_t u\|^2_{L^2} + \|\nabla_x u\|^2_{L^2} + \|u\|^{q+2}_{L^{q+2}} + \left\|\phi(\nabla_x u)\right\|_{L^1}.$$

The next lemma gives the formal dissipative estimate for the H^2-norm of a strong solution u.

Lemma 4.1. *Let the above assumptions hold and let u be a sufficiently regular solution of problem* (1.1)*. Then, the following estimate holds*:

$$\begin{aligned} &\left\|u(t)\right\|^2_{W^{2,2}} + \int_t^{t+1} \left(\left|\phi''\left(\nabla_x u(s)\right)\right|, \left|D^2_x u(s)\right|^2\right) ds\\ &\quad \leqslant Q\left(\left\|\xi_u(0)\right\|_{\mathcal{E}_\phi} + \left\|u(0)\right\|_{W^{2,2}}\right)e^{-\beta t} + Q\left(\|g\|_{L^2}\right), \end{aligned} \tag{4.2}$$

where $D^2_x u$ means a collection of all second derivatives $\partial_{x_i}\partial_{x_j}u$ of the function u and the monotone function Q and the exponent $\beta>0$ depend only on the constants κ_i and from (4.1) (*and independent of the concrete choice of ϕ satisfying these conditions*).

Proof. The key idea here is to multiply (analogously to the semi-linear case, see, e.g., [37,50]) Eq. (1.1) by $\Delta_x u$ and the main difficulty is the quasi-linear term. In the case of *periodic* boundary conditions not any additional problems arise, since

$$\begin{aligned} \sum_i\left(\nabla_x\phi'(\nabla_x u), \partial^2_{x_i}u\right) &= \sum_{i,j}\left(\partial_{x_i}\phi'_j(\nabla_x u), \partial_{x_i}\partial_{x_j}u\right)\\ &= \sum_i\left(\phi''(\nabla_x u)\partial_{x_i}\nabla_x u, \partial_{x_i}\nabla_x u\right) \geqslant \kappa\left(\left|\phi''(\nabla_x u)\right|, \left|D^2_x u\right|^2\right), \end{aligned} \tag{4.3}$$

where we have used the first inequality of (4.1). However, this does not work directly for the case of Dirichlet boundary conditions due to the appearance of uncontrollable boundary terms under the integration by parts.

In order to overcome this difficulty, we will use some kind of localization technique. First, we deduce the H^2-estimate *inside* of the domain Ω. To this end, we multiply Eq. (1.1) by $\nabla_x(\theta(x)\nabla_x u)$ where $\theta(x)$ is a cut-off function which equals zero near the boundary and one in the δ-interior of the domain Ω and satisfies the inequality

$$\left|\theta''(x)\right| + \left|\theta'(x)\right| \leqslant C\left[\theta(x)\right]^{1/2}.$$

Then, analogously to (4.3), we will have

$$\begin{aligned}\big(\nabla_x\phi'(\nabla_x u),\nabla_x(\theta\nabla_x u)\big)&=\sum_{i,j}\big(\partial_{x_i}\phi'_j(\nabla_x u),\partial_{x_i}(\theta\partial_{x_j}u)\big)\\&\geqslant\kappa\big(\theta\big|\phi''(\nabla_x u)\big|,\big|D^2_x u\big|^2\big)-C\big(\big|\phi'(\nabla_x u)\big|,\big(|\theta''|+|\theta'|\big)\big|D^2_x u\big|\big)\\&\geqslant 1/2\kappa\big(\theta\big|\phi''(\nabla_x u)\big|,\big|D^2 u\big|^2\big)-C_1\big(1+\big\|\phi(\nabla_x u)\big\|_{L^1}\big).\end{aligned}\tag{4.4}$$

Here we have used assumption (4.1)(4) and the following estimate

$$\begin{aligned}&\big(\big|\phi'(\nabla_x u)\big|,\big(|\theta''|+|\theta'|\big)|w|\big)\\&=\big(\big|\phi''(\nabla_x v)\big|^{-1/2}\big|\phi'(\nabla_x u)\big|,\big(|\theta''|+|\theta'|\big)\big|\phi''(\nabla_x u)\big|^{1/2}\cdot|w|\big)\\&\leqslant\big(\big|\phi''(\nabla_x u)\big|^{-1}\cdot\big|\phi'(\nabla_x u)\big|^2,1\big)^{1/2}\big(\big(|\theta''|^2+|\theta'|^2\big)\big|\phi''(\nabla_x u)\big|,|w|^2\big)^{1/2}\\&\leqslant 1/2\kappa\big(\theta\big|\phi''(\nabla_x u)\big|,|w|^2\big)+C\big(1+\big\|\phi(\nabla_x u)\big\|_{L^1}\big).\end{aligned}\tag{4.5}$$

The other terms are much simpler to estimate. For instance, due to the quasi-monotonicity assumption $f'(u)\geqslant -K$, the other non-linear term factually disappears

$$-\big(f(u),\nabla_x(\theta\nabla_x u)\big)=\big(\theta f'(u)\nabla_x u,\nabla_x u\big)\geqslant -K\|\nabla_x u\|^2_{L^2}$$

and the linear terms can be estimated as follows:

$$\begin{aligned}\big(\partial_t\Delta_x u,\nabla_x(\theta\nabla_x u)\big)&=\frac{1}{2}\partial_t\big(\theta,|\Delta_x u|^2\big)-\big(\partial_t\nabla_x u,\nabla_x(\theta'\nabla_x u)\big)\\&\geqslant\frac{1}{2}\partial_t\big(\theta,|\Delta_x u|^2\big)-C_\varepsilon\|\partial_t\nabla_x u\|^2-\varepsilon\big(\theta,|\Delta_x u|^2\big)-C\|\nabla_x u\|^2_{L^2}\end{aligned}\tag{4.6}$$

and

$$\begin{aligned}-\big(\partial_t^2 u,\nabla_x(\theta\nabla_x u)\big)&=\big(\partial_t^2\nabla_x u,\theta\nabla_x u\big)=\partial_t(\partial_t\nabla_x u,\theta\nabla_x u)-(\theta\partial_t\nabla_x u,\partial_t\nabla_x u)\\&\geqslant-\partial_t\big(\partial_t u,\nabla_x(\theta\nabla_x u)\big)-C\|\partial_t\nabla_x u\|^2.\end{aligned}\tag{4.7}$$

Thus, combining the above estimates, we arrive at

$$\begin{aligned}&\frac{d}{dt}\bigg[\frac{\gamma}{2}\big(\theta,|\Delta_x u|^2\big)-\big(\partial_t u,\nabla_x(\theta\nabla_x u)\big)\bigg]+\big[\big(\theta,|\Delta_x u|^2\big)-\big(\partial_t u,\nabla_x(\theta\nabla_x u)\big)\big]\\&\quad+\frac{1}{2}\kappa\big(\theta\big|\phi''(\nabla_x u)\big|,\big|D^2_x u\big|^2\big)\\&\leqslant C\big(1+\big\|\phi(\nabla_x u)\big\|_{L^1}+\|\partial_t\nabla_x u\|^2_{L^2}+\|\nabla_x u\|^2_{L^2}\big).\end{aligned}\tag{4.8}$$

Applying the Gronwall inequality to this relation and using the following analog of the dissipative estimate (2.1)

$$\begin{aligned}&\big\|\partial_t u(t)\big\|^2_{L^2}+\big\|u(t)\big\|^2_{H^2}+\big\|\phi\big(\nabla_x u(t)\big)\big\|_{L^1}+\big\|u(t)\big\|^{q+2}_{L^{q+2}}+\int_t^{t+1}\big\|\partial_t\nabla_x u(s)\big\|^2_{L^2}\,ds\\&\quad\leqslant Q\big(\big\|\xi_u(0)\big\|_{\mathcal{E}_\phi}\big)e^{-\gamma t}+C\big(1+\|g\|^2_{L^2}\big)\end{aligned}\tag{4.9}$$

in order to control the right-hand side of (4.8), we deduce the required H^2-estimate inside of the domain Ω:

$$\|\theta u(t)\|^2_{W^{2,2}} + \int_t^{t+1} \big(\theta\big|\phi''\big(\nabla_x u(s)\big)\big|, \big|D^2_x u(s)\big|^2\big)\,ds$$
$$\leqslant Q\big(\big\|\xi_u(0)\big\|_{\mathcal{E}_\phi} + \big\|u(0)\big\|_{W^{2,2}}\big)e^{-\beta t} + Q\big(\|g\|_{L^2}\big). \tag{4.10}$$

At the next step, we consider the neighborhood of the boundary $\partial\Omega$. To this end, we extend the tangent ($\tau_1 = (\tau_1^1, \tau_2^2, \tau_1^3)$ and $\tau_2 = (\tau_2^1, \tau_2^2, \tau_2^3)$) and normal ($n = (n^1, n^2, n^3)$) vector fields from the boundary $\partial\Omega$ inside of the domain Ω in such a way that the basis $(\tau_1(x), \tau_2(x), n(x))$ will be orthonormal at least in a small neighborhood of the boundary (being pedantic, in a small neighborhood of an arbitrarily fixed point $x_0 \in \partial\Omega$; outside of this neighborhood, the above vector fields should be cutted off using, e.g., the scalar cut-off multiplier $\theta(x)$). Let $\partial_{\tau_i} := \sum_{j=1}^3 \tau_i^j(x)\partial_{x_j}$ and $\partial_n := \sum_{j=1}^3 n^j(x)\partial_{x_j}$ be the associated differential operators. Let also $\partial^*_{\tau_i}$, $i = 1, 2$, be the formally adjoint (in $L^2(\Omega)$) differential operators, i.e.,

$$\big(\partial^*_{\tau_i} v\big)(x) := -\sum_{j=1}^3 \partial_{x_j}\big(\tau_i^j(x)v(x)\big).$$

We now multiply the equation by the term

$$-\partial^*_{\tau_i}\partial_{\tau_i} u = \sum_{j,k} \partial_{x_k}\big(\tau_i^k(x)\tau_i^j(x)\partial_{x_j} u\big).$$

Then, since $D_\tau u|_{\partial\Omega} = 0$ (here and below, D_τ stands for the one of tangent derivatives ∂_{τ_1} or ∂_{τ_2}), we do not have any boundary terms under the integrating by parts in the quasi-linear term. Moreover, we also have

$$|\partial_{\tau_i}\nabla_x F - \nabla_x\partial_{\tau_i} F| \leqslant C|\nabla_x F|.$$

By this reason, analogously to (4.4) and (4.5), we can estimate the quasi-linear term by

$$\begin{aligned}
-\big(\nabla_x\phi'(\nabla_x u), D^*_\tau D_\tau u\big) &= -\big(D_\tau\nabla_x\phi'(\nabla_x u), D_\tau u\big)\\
&\geqslant -\big(\nabla_x D_\tau\phi'(\nabla_x u), D_\tau u\big) - C\big(\big|\phi'(\nabla_x u)\big|, |\nabla_x D_\tau u|\big)\\
&= \big(D_\tau\phi'(\nabla_x u), \nabla_x D_\tau u\big) - C\big(\big|\phi'(\nabla_x u)\big|, |\nabla_x D_\tau u|\big)\\
&\geqslant \kappa\big(\big|\phi''(\nabla_x u)\big|, |\nabla_x D_\tau u|^2\big) - C\big(\big|\phi'(\nabla_x u)\big|, |\nabla_x D_\tau u|\big)\\
&\geqslant 1/2\kappa\big(\big|\phi''(\nabla_x u)\big|, |\nabla_x D_\tau u|^2\big) - C_1\big(1 + \big\|\phi(\nabla_x u)\big\|_{L^1}\big).
\end{aligned} \tag{4.11}$$

The linear terms are easier to estimate, in particular,

$$-\big(\partial_t\Delta_x u, D^*_\tau D_\tau u\big) \geqslant \frac{1}{2}\frac{d}{dt}\|\nabla_x D_\tau\|^2_{L^2} - C\|\partial_t\nabla_x u\|_{L^2}\|\nabla_x D_\tau u\|_{L^2}.$$

Then, arguing exactly as in the derivation of (4.10), we obtain the control of second derivatives of u in tangential directions:

$$\left\|\nabla_x D_\tau u(t)\right\|^2_{L^2} + \int_t^{t+1} \left(\left|\phi''\left(\nabla_x u(s)\right)\right|, \left|\nabla_x D_\tau u(s)\right|^2\right) ds$$
$$\leqslant Q\left(\left\|\xi_u(0)\right\|_{\mathcal{E}_\phi} + \left\|u(0)\right\|_{W^{2,2}}\right)e^{-\beta t} + Q\left(\|g\|_{L^2}\right). \tag{4.12}$$

Thus, we only need to estimate the second normal derivatives. To this end, we multiply Eq. (1.1) by $\Delta_x u$ and, instead of integrating by parts in the quasi-linear term, will now use the fact that all of the second derivatives except of $\partial_n^2 u$ are already under the control (due to (4.12) and (4.10)). More precisely, let $n := \tau_3$ and

$$\partial_{x_i} = A_{ik}(x)\partial_{\tau_k}, \quad i,k = 1,2,3.$$

Then, since the vector fields (τ_1, τ_2, n) are orthonormal (at least near the boundary), we may conclude that

$$\left|\Delta_x u - \partial_n^2 u\right| \leqslant C\left(|\nabla_x D_\tau|^2 + |\nabla_x u|^2 + |u|^2 + \left|\theta D_x^2 u\right|^2\right),$$

where θ is the cut-off function defined above. Moreover, analogously

$$\left|\nabla_x \phi'(\nabla_x u) - \left(\sum_{ij} \phi''_{ij}(\nabla_x u) A_{i3} A_{j3}\right)\partial_n^2 u\right|$$
$$\leqslant C\left(\left|\phi''(\nabla_x u)\right| \cdot |\nabla_x D_\tau u|^2 + \theta\left|\phi''(\nabla_x u)\right| \cdot \left|D_x^2 u\right|^2 + \left|\phi''(\nabla_x u)\right| \cdot |\nabla_x u|^2\right)$$

and, due to (4.1)(1)

$$\sum_{ij} \phi''_{ij}(\nabla_x u) A_{i3} A_{j3} \geqslant \kappa\left|\phi''(\nabla_x u)\right|$$

(at least in the small neighborhood of the boundary). Using that estimates, we finally infer

$$\left(\nabla_x \phi'(\nabla_x u), \Delta_x u\right) \geqslant \kappa\left(\left|\phi''(\nabla_x u)\right|, \left|\partial_n^2 u\right|^2\right) - C\left(\left|\phi''(\nabla_x u)\right|, |\nabla_x D_\tau u|^2\right) - C\left(1 + \left\|\phi(\nabla_x u)\right\|_{L^1}\right)$$
$$- C\left(\theta\left|\phi''(\nabla_x u)\right|, \left|D_x^2 u\right|^2\right). \tag{4.13}$$

Using now estimates (4.12) and (4.10) together with (4.9) in order to control the subordinated terms, we get

$$\left\|\Delta_x u(t)\right\|^2_{L^2} + \int_t^{t+1} \left(\left|\phi''\left(\nabla_x u(s)\right)\right|, \left|\partial_n^2 u(s)\right|^2\right) ds \leqslant Q\left(\left\|\Delta_x u(0)\right\|_{L^2} + \left\|\xi_u(0)\right\|_{\mathcal{E}_\phi}\right)e^{-\beta t} + C\left(1 + \|g\|^2_{L^2}\right)$$

which together with (4.12) and (4.10) give the desired estimate (4.2) and finish the proof of the lemma. □

We are now ready to prove the existence of H^2-solutions. In contrast to the previous section, the Galerkin method seems not applicable here since the multiplication of the equation by $\partial_\tau^* \partial_\tau u$ will be problematic on the level of Galerkin approximations. By this reason, we will proceed in an alternative way approximating the non-linearity ϕ by the sequence ϕ_n such that ϕ''_n are globally bounded. For this reason, we needed the modified assumptions (4.1). We formulate the main result of this section in the following theorem.

Theorem 4.2. *Let the non-linearities ϕ and f satisfy* (1.3) (*with arbitrary growth exponents p and q!*)*. Then, for every $\xi_u(0) \in \mathcal{E} = \mathcal{E}(p,q)$ such that $u(0) \in H^2(\Omega)$, there exists at least one weak energy solution $\xi_u(t)$ with the additional regularity $u(t) \in H^2(\Omega)$ which satisfies the following estimate*:

$$\|\xi_u(t)\|_{\mathcal{E}} + \|u(t)\|_{H^2} \leqslant Q\left(\|\xi_u(0)\|_{\mathcal{E}} + \|u(0)\|_{H^2}\right)e^{-\beta t} + Q\left(\|g\|_{L^2}\right) \tag{4.14}$$

for some positive constant β and monotone function Q.

Proof. The proof of solvability is based on the following observation: if the non-linearity $\phi(v)$ is such that $\phi''(v)$ is globally bounded and the initial data is smooth enough, say $\xi_u(0) \in \mathcal{E}_1$, then, for any H^2-solution $u(t)$ all of the terms in Eq. (1.1) belong to $L^2([0,T] \times L^2(\Omega))$. Indeed, since $\phi''(v) \leqslant C$, then $u \in H^2$ implies that $\nabla_x \phi'(\nabla_x u) \in L^2$ and $\partial_t^2 u \in L^2([0,T] \times \Omega)$ due to Proposition 2.4. Thus, all terms except of $\partial_t \Delta_x u$ in (1.1) belong indeed to L^2 and, therefore, the term $\partial_t \Delta_x u$ should also belong to that space.

By this reason, we are able to multiply Eq. (1.1) by $\Delta_x u$ and $\partial_\tau^* \partial_\tau u$ and, therefore, the formal computations can be justified in a standard way if, in addition,

$$\left|\phi''(v)\right| \leqslant C, \qquad \xi_u(0) \in \mathcal{E}_1. \tag{4.15}$$

Thus, we will approximate the non-linearity $\phi(v)$ by a sequence $\{\phi_n(v)\}$ which is globally bounded in such a way that assumptions (4.1) hold *uniformly* with respect to $n \to \infty$. Although it is more or less clear that such a sequence exists, for the convenience of the reader we will briefly describe below one possible way to construct it.

At the first step, we note that such a sequence obviously exists in the particular case

$$\phi_r(v) := |v|^r$$

for any fixed $r \geqslant 2$. For instance, we can take $\phi_N(v) := \theta_N(\phi(v))$ where the cut-off function is such that

$$\theta_N(x) = \begin{cases} x, & \text{for } x \leqslant N; \\ A_N + B_N x^{2/r}, & \text{for } x \geqslant N, \end{cases}$$

where the coefficients A_N and B_N are such that $\theta_N \in C^1$ near $x = N$ (being pedantic, the constructed function has a jump of the second derivative at $x = N$, but it is not essential since all the above estimates work for such type of non-linearities ϕ).

At the second step, we introduce a family of functions

$$\phi_\varepsilon(v) := \phi(v) + \varepsilon\phi_r(v),$$

where the exponent r is *larger* than $p+1$ (recall that the non-linearities ϕ and f satisfy (1.2) and (1.3)) and observe that assumptions (4.1) hold for the functions $\phi_\varepsilon(v)$ *uniformly* with respect to $\varepsilon \to 0$.

Finally, at the third step we construct the desired approximations

$$\phi_n(v) := \theta_{N_n}\left(\phi(v) + \varepsilon_n \phi_r(v)\right),$$

where the sequence $\varepsilon_n \to 0$ (which guarantees the uniform convergence of ϕ_n to ϕ at every compact set) and the sequence N_n tends to infinity fast enough in order to guarantee that the cut-off starts at the region where

$$\varepsilon_n \phi_r(v) \gg \phi(v).$$

Therefore, in that region, the leading term of $\phi_n(v)$ is $\theta_{N_n}(\varepsilon_n\phi_r(v))$ for which (4.1) are satisfied *uniformly* with respect to n. By this reason, these assumptions will be satisfied also for the perturbed functions $\phi_n(v)$ and also uniformly with respect to $n \to \infty$.

Thus, we have constructed the family $\phi_n(v)$ with the following properties:

(1) ϕ_n satisfy (4.1) uniformly with respect to $n \to \infty$;
(2) $\phi_n(v) \to \phi(v)$ uniformly with respect to all $v \in \mathbb{R}^3$ such that $|v| \leqslant R$ (R is arbitrary);
(3) the following additional inequalities hold:

$$1.\ \big|\phi_n(v)\big| \leqslant C\big(1+|v|^r\big), \qquad 2.\ \big|\phi_n'(v)\big| \leqslant C\big(1+\phi(v)\big)^{r/(r+1)}, \tag{4.16}$$

where the constant C is independent of n.

We are now ready to verify the existence of the desired solution. To this end, we first consider the case of regular initial data $\xi_u(0) \in C^2(\Omega) \times C^2(\Omega)$ and construct a (unique) approximate solution u_n by solving the following system:

$$\partial_t^2 u_n - \gamma\partial_t\Delta_x u_n - \Delta_x u_n + f(u_n) = \nabla_x\phi_n'(\nabla_x u_n) + g, \qquad \xi_{u_n}(0) = \xi_u(0). \tag{4.17}$$

Then, due to Lemma 4.1, we have the uniform (with respect to n) estimate

$$\big\|u_n(t)\big\|^2_{W^{2,2}} \leqslant Q\left(\big\|\xi_u(0)\big\|_{\mathcal{E}_{\phi_n}} + \big\|u(0)\big\|_{W^{2,2}}\right)e^{-\beta t} + Q\big(\|g\|_{L^2}\big). \tag{4.18}$$

In particular, since $\xi_u(0)$ is smooth, the first estimate in (4.16) guarantees that

$$\big\|\xi_u(0)\big\|_{\mathcal{E}_{\phi_n}} \to \big\|\xi_u(0)\big\|_{\mathcal{E}_{\phi}}.$$

It is now not difficult to pass to the limit in Eqs. (4.17). Indeed, due to the uniform estimate (4.2), we may assume without loss of generality that ξ_{u_n} converges weakly-star to some function ξ_u in the space

$$L^\infty\big([0,T],\big[H^2(\Omega)\cap H_0^1(\Omega)\big]\times L^2(\Omega)\big)$$

and we need to prove that u solves (1.1) by passing to the limit in (4.17). As usual, the passage to the limit in the linear terms are immediate and we only need to take care on the non-linear ones. Note that the above weak convergence implies the strong convergence

$$u_n \to u \quad \text{in the space } C\big([0,T],H^{2-\varepsilon}(\Omega)\big) \subset C\big([0,T]\times\Omega\big) \tag{4.19}$$

and, therefore, $f(u_n) \to f(u)$. So, we only need to check that

$$\nabla_x\phi_n'(\nabla_x u_n) \to \nabla_x\phi'(\nabla_x u)$$

in the sense of distributions. To this end, we note that, due to (4.16),

$$\big\|\phi'(\nabla_x u)\big\|_{L^{(r+1)/r}} \leqslant C.$$

Moreover, from (4.19) we conclude that $\nabla_x u_n \to \nabla_x u$ almost everywhere in $[0,T]\times\Omega$ and, therefore, $\phi_n'(\nabla_x u_n) \to \phi'(\nabla_x u)$ almost everywhere. Thus, we have proved that $\phi_n'(\nabla_x u_n) \to \phi'(\nabla_x u)$ in $L^{(r+1)/r}$ and that u solves indeed Eq. (1.1) in the sense of distributions.

Let us check that the solution u belongs to $L^\infty([0,T],W^{1,p+1}(\Omega))$ and satisfies the dissipative estimate (4.14). To this end, it is sufficient to note that, due to the convergence $\phi_n(\nabla_x u_n)\to\phi(\nabla_x u)$ almost everywhere, the Fatou lemma gives

$$\big\|\phi(\nabla_x u)\big\|_{L^1}\leqslant \liminf_{n\to\infty}\big\|\phi_n(\nabla_x u)\big\|_{L^1} \tag{4.20}$$

which allows to verify (4.14) by passing to the limit $n\to\infty$ in estimates (4.18) (exactly, in order to pass to the limit in the right-hand side, we need the additional assumption on the initial data to be smooth).

Thus, we have constructed the desired H^2-solution of problem (1.1) under the additional assumption that

$$\big(u(0),\partial_t u(0)\big)\in C^2(\Omega)\times C^2(\Omega).$$

Finally, this assumption can be easily removed by approximating the arbitrary H^2-initial data by the smooth one and passing to the limit once more. Theorem 4.2 is proved. □

Remark 4.3. Passing to the limit $n\to\infty$ in the dissipative estimate (4.2) for the approximations u_n, we can prove that the limit solution $u(t)$ also satisfies

$$\int_t^{t+1}\big(\big|\phi''\big(\nabla_x u(s)\big)\big|,\big|D_x^2u(s)\big|^2\big)\,ds\leqslant Q\big(\big\|\xi_u(0)\big\|_{\mathcal{E}}+\big\|\Delta_x u(0)\big\|_{L^2}\big)e^{-\gamma t}+Q\big(\|g\|_{L^2}\big) \tag{4.21}$$

which, together with our growth assumptions on ϕ and Sobolev embedding theorem, gives the following control

$$\|\nabla_x u\|_{L^{p+1}([T,T+1],L^{3(p+1)}(\Omega))}\leqslant Q\big(\big\|\xi_u(0)\big\|_{\mathcal{E}}+\big\|\Delta_x u(0)\big\|_{L^2}\big)e^{-\gamma t}+Q\big(\|g\|_{L^2}\big) \tag{4.22}$$

which is important for establishing the uniqueness of strong solutions for the limit growth exponent $p=5$, see below.

Note once more that the above proved theorem does not require any growth restrictions on ϕ. However, in that case, we cannot guarantee the uniqueness. Combining now Proposition 2.4 with Theorem 4.2, we obtain the following result on the well-posedness and dissipativity of strong solutions.

Corollary 4.4. *Let the assumptions of Theorem* 2.2 *hold* (*in particular $p<5$ now*)*. Then, for every $\xi_u(0)\in\mathcal{E}_1$* (*see* (1.19) *for the definition*)*, there exists a unique strong solution $u(t)$ of problem* (1.1) *and the following estimate holds*:

$$\begin{aligned}&\big\|\partial_t u(t)\big\|_{H^1}^2+\big\|\partial_t^2u(t)\big\|_{H^{-1}}^2+\big\|\nabla_x\phi'\big(\nabla_x u(t)\big)\big\|_{H^{-1}}^2+\big\|u(t)\big\|_{H^2}^2\\&\quad\leqslant Q\big(\big\|\xi_u(0)\big\|_{\mathcal{E}_1}\big)e^{-\beta t}+Q\big(\|g\|_{L^2}\big)\end{aligned} \tag{4.23}$$

for some positive constant β and monotone function Q.

Indeed, estimate (4.23) is an immediate corollary of (2.19) and (4.14) (the assumption $\nabla_x\phi'(\nabla_x u(0))\in H^{-1}$ guarantees that $\partial_t^2u(0)\in H^{-1}$ and Proposition 2.4 is indeed applicable; vice versa, the control of the H^{-1}-norm of $\partial_t^2 u$ obtained from Proposition 2.4 together with the H^2-estimate allows to control the H^{-1}-norm of $\nabla_x\phi'(\nabla_x u(t))$).

Remark 4.5. We see that the additional non-linear condition $\nabla_x\phi'(\nabla_x u) \in H^{-1}$ is included to the definition (1.19) of the space $\mathcal{E}_1$ in order to be able to control the H^{-1}-norm of the second time derivative $\partial_t^2 u(t)$ of the solution (which is crucial for our method). In the case $p \leqslant 3$, the assumption $u \in H^2$ automatically implies that $\phi'(\nabla_x u) \in L^2(\Omega)$ and, therefore, $\nabla_x\phi'(\nabla_x u) \in H^{-1}$. So, this additional condition is not necessary if $p \leqslant 3$ and the strong energy space $\mathcal{E}_1$ has a usual form

$$\mathcal{E}_1 = \left[H^2 \cap H_0^1\right] \times H_0^1.$$

However, it is not so if the non-linearity ϕ grows faster ($p > 3$) since in that case the $W^{1,2p}$-norm of the solution is no more controlled by its H^2-norm and the assumption $\nabla_x\phi'(\nabla_x u) \in H^{-1}$ gives the additional information about the regularity of the solution which cannot be obtained from the linear terms. Indeed, assume that the quasi-linear elliptic problem

$$\nabla_x \cdot \phi'(\nabla_x U) = h, \qquad U|_{\partial\Omega} = 0, \quad h \in H^{-1}(\Omega), \tag{4.24}$$

possesses the maximal regularity theorem in the form

$$\left\|\phi'(\nabla_x U)\right\|_{L^2} \leqslant Q\left(\|h\|_{H^{-1}}\right) \tag{4.25}$$

for every $h \in H^{-1}$ and some monotone function Q (for the particular case $\phi(z) := |z|^{p+1}$, this regularity result is known to be true, see [1]). In that case, due to our growth assumptions on ϕ, the additional condition $\nabla_x\phi'(\nabla_x u) \in H^{-1}$ is simply equivalent to $u \in W^{1,2p}(\Omega)$ and the strong energy space $\mathcal{E}_1$ now reads

$$\mathcal{E}_1 = \left[H^2(\Omega) \cap W_0^{1,2p}(\Omega)\right] \times H_0^1(\Omega). \tag{4.26}$$

Unfortunately, we do not know whether or not the regularity estimate (4.25) holds for the general non-linearity ϕ satisfying (1.2) and, by this reason, have to write the strong energy space $\mathcal{E}_1$ in the form of (1.19).

To conclude the section, we briefly discuss how to handle the limit case $p = 5$. In that case, for obtaining the uniqueness, we need to include the control $\nabla_x u \in L^6([0,T], L^{18}(\Omega))$ (see (4.22)) into the definition of a strong solution.

Proposition 4.6. *Let the assumptions* (1.2) *and* (1.3) *be satisfied with* $p = 5$. *Then, the strong solution* $u(t)$ *of problem* (1.1) *which satisfies the additional regularity*

$$\nabla_x u \in L^6\left([0,T], L^{18}(\Omega)\right) \tag{4.27}$$

is unique.

Proof. Indeed, in the proof of Theorem 2.2, we have used the assumption $p < 5$ in order to estimate the term

$$\left(\left[\phi'(\nabla_x u_1) - \phi'(\nabla_x u_2)\right], \nabla_x(-\Delta_x)^{-1}\partial_t v\right)$$

only. Instead, we now estimate this term based on (4.27) and using the Hölder inequality with exponents 3, 2 and 6 as follows:

$$\left(\int_0^1 \phi''\big(s\nabla_x u_1 + (1-s)\nabla_x u_2\big)\,ds\,\nabla_x v,\ \nabla_x(-\Delta_x)^{-1}\partial_t v\right)$$
$$\leqslant C\big(1 + \|\nabla_x u_1\|_{L^{12}} + \|\nabla_x u_2\|_{L^{12}}\big)^4 \|\nabla_x v\|_{L^2}\|\partial_t v\|_{L^2}$$
$$\leqslant \varepsilon\|\partial_t v\|^2_{L^2} + C_\varepsilon\big(1 + \|\nabla_x u_1\|_{L^{12}} + \|\nabla_x u_2\|_{L^{12}}\big)^8 \|\nabla_x v\|^2_{L^2}. \tag{4.28}$$

Since, due to the interpolation inequality,

$$L^\infty\big([0,T], H^1\big) \cap L^6\big([0,T], L^{18}(\Omega)\big) \subset L^8\big([0,T], L^{12}(\Omega)\big),$$

assumption (4.27) together with the fact that the H^2-norm of $u(t)$ is bounded, gives the control

$$\int_0^T \big\|\nabla_x u(s)\big\|^8_{L^{12}}\,ds \leqslant C_T.$$

Thus, analogously to (2.12), we get

$$\partial_t\big[\gamma/2\|\nabla_x v\|^2_{L^2} + (\partial_t v, v) + 2/\gamma\big(\|\partial_t v\|^2_{H^{-1}} + \|v\|^2_{L^2}\big)\big] + \beta\big(\|\partial_t v\|^2_{L^2} + \|\nabla_x v\|^2_{H^1}\big)$$
$$\leqslant C\big(1 + \|\nabla_x u_1\|_{L^{12}} + \|\nabla_x u_2\|_{L^{12}}\big)^8\big(\|\nabla_x v\|^2 + \|\partial_t v\|^2_{H^{-1}}\big). \tag{4.29}$$

Applying the Gronwall inequality to this relation, we will have

$$\big\|\partial_t v(t)\big\|^2_{H^{-1}} + \big\|v(t)\big\|^2_{H^1} \leqslant C(t)\big(\big\|\partial_t v(0)\big\|^2_{H^{-1}} + \big\|v(0)\big\|^2_{H^1}\big) \tag{4.30}$$

which gives the desired uniqueness. □

Corollary 4.7. *Let the assumptions of Proposition* 4.6 *hold. Then, estimate* (2.19) *is satisfied.*

Proof. Indeed, arguing as in the proof of Proposition 2.4, but using estimate (4.28), we derive the analog of (2.19), but with the *growing* in time coefficients, see (4.29) and (4.30).

In order to overcome this difficulty, we need to combine this estimate with the smoothing property of Proposition 2.5 which can be obtained by multiplying inequality (4.29) (of course, with $v = \partial_t u$) by t^3 and arguing exactly as in (2.26). This finishes the proof of the corollary. □

Corollary 4.8. *Let the assumptions of Proposition* 4.6 *be satisfied and let $\xi_u(0) \in \mathcal{E}_1$. Then, the following dissipative estimate holds for the unique strong solution $u(t)$ of problem* (1.1)

$$\big\|u(t)\big\|^2_{H^2} + \big\|\partial_t u(t)\big\|^2_{H^1} + \big\|\partial_t^2 u(t)\big\|^2_{H^{-1}} + \int_t^{t+1} \big\|\nabla_x u(s)\big\|^6_{L^{18}}\,ds$$
$$\leqslant Q\big(\big\|\xi_u(0)\big\|_{\mathcal{E}_1}\big)e^{-\beta t} + Q\big(\|g\|_{L^2}\big) \tag{4.31}$$

for some positive constant β and monotone function Q.

Indeed, the above estimate is a combination of (4.14), (4.22) and (2.19).

5. Regularity of the global attractor and the exponential attractor

In this section, we verify that the weak global attractor $\mathcal{A}$ consists of strong solutions and construct an exponential attractor for the semigroup $\{S(t), t \geqslant 0\}$ associated with Eq. (1.1) in the energy space $\mathcal{E} = \mathcal{E}(p,q)$. We start with the following theorem which establishes the asymptotic regularity of that semigroup.

Theorem 5.1. *Let the assumptions of Theorem* 2.2 *hold. Then, for every bounded set B in $\mathcal{E}(p,q)$, the following estimate is satisfied*:

$$\operatorname{dist}_{\mathcal{E}_{-1}}\big(S(t)B, \big\{\xi \in \mathcal{E}_1,\ \|\xi\|_{\mathcal{E}_1} \leqslant K\big\}\big) \leqslant Q\left(\|B\|_{\mathcal{E}}\right)e^{-\beta t} \tag{5.1}$$

if K is large enough. Here β is some positive number, Q is a monotone function both independent of B and t and $\operatorname{dist}_V(X,Y)$ is a non-symmetric Hausdorff distance between sets in V.

Proof. Due to the dissipative estimate (2.1), it is sufficient to prove the theorem for the absorbing ball $B = B_R$.

Let now $u(t)$ be any trajectory starting from the absorbing ball B_R. Then, due to Proposition 2.5, we may assume without loss of generality that

$$\int_t^{t+1} \left\|\partial_t^2 u(s)\right\|^2 ds + \left\|\partial_t^2 u(t)\right\|_{H^{-1}}^2 + \left\|\partial_t u(t)\right\|_{H^1}^2 \leqslant C_R, \tag{5.2}$$

where the constant C_R is independent of t. Thus, it is sufficient to verify that

$$\operatorname{dist}_{H^1}\big(u(t), \big\{u_0 \in H^2,\ \|u_0\|_{H^2} \leqslant K\big\}\big) \leqslant C_R e^{-\beta t}. \tag{5.3}$$

In order to do so we split the function u as follows: $u(t) = v(t) + w(t)$ where the function w solves

$$\begin{gathered} -\gamma\partial_t\Delta_x w - \Delta_x w + Lw + f(w) - \nabla_x\big(\phi'(\nabla_x w)\big) = h_u(t) := -\partial_t^2 u(t) + g + Lu(t), \\ w|_{t=0} = 0, \end{gathered} \tag{5.4}$$

where L is a sufficiently large number which will be fixed below. Then, the reminder $v(t)$ should solve

$$\begin{gathered} -\gamma\partial_t\Delta_x v - \Delta_x v + \big[f(v+w) - f(w)\big] + Lv = \nabla_x\big(\phi'(\nabla_x v + w) - \phi'(\nabla_x w)\big), \\ v|_{t=0} = w|_{t=0}. \end{gathered} \tag{5.5}$$

The desired result will follow from two lemmas formulated below.

Lemma 5.2. *Let the above assumptions hold. Then the solution $w(t)$ belongs to H^2 for every $t \geqslant 0$ and the following estimate holds*:

$$\left\|w(t)\right\|_{H^2} \leqslant C_L, \tag{5.6}$$

where the constant C_L may depend on L, but is independent of the concrete choice of the trajectory u starting from the absorbing ball B_R and satisfying (5.2).

Proof. Indeed, due to (5.2), we have the L^2-control of the right-hand side of (5.4):

$$\int_t^{t+1} \|h_u(s)\|_{L^2}^2\,ds \leqslant C_L. \tag{5.7}$$

This, allows us to verify estimate (5.6) just repeating word by word the proof of Lemma 4.1. By this reason, we leave the proof to the reader. □

Lemma 5.3. *Let the above assumptions hold and let L be large enough. Then, the v component of the solution u satisfies the following estimate*:

$$\|v(t)\|_{H^1} \leqslant \|u(0)\|_{H^1} e^{-\beta t} \tag{5.8}$$

for some positive exponent β.

Proof. Indeed, multiplying Eq. (5.5) by v, integrating over Ω and using the monotonicity of ϕ', we get

$$\frac{\gamma}{2}\frac{d}{dt}\|\nabla_x v(t)\|^2 + \|\nabla_x v(t)\|_{L^2}^2 + \big(f\big(v(t)+w(t)\big) - f\big(w(t)\big), v(t)\big) + L\|v(t)\|_{L^2}^2 \leqslant 0.$$

Using now assumption (1.3), we see that

$$\big(f\big(v(t)+w(t)\big) - f\big(w(t)\big), v(t)\big) \geqslant -C\|v(t)\|_{L^2}^2$$

and, therefore, the Gronwall inequality gives the desired estimate (5.8) if $L \geqslant C$. □

Combining estimates (5.8) and (5.6), we deduce (5.3) and finish the proof of the theorem. □

As an immediate corollary of the proved theorem, we obtain the following result on the regularity of the weak global attractor $\mathcal{A}$.

Corollary 5.4. *Let the assumptions of Theorem* 2.2 *hold. Then, the weak global attractor $\mathcal{A}$ of the solution semigroup $\{S(t),\ t \geqslant 0\}$ associated with Eq.* (1.1) *belongs to the space $\mathcal{E}_1$ and is bounded there*:

$$\|\mathcal{A}\|_{\mathcal{E}_1} \leqslant C. \tag{5.9}$$

In particular, the attractor $\mathcal{A}$ is generated by strong solutions of problem (1.1).

We are now ready to overcome the difficulty mentioned in Remark 3.3 and to construct an exponential attractor for the solution semigroup $\{S(t),\ t \geqslant 0\}$. We first recall the exponential attractor definition adapted to our situation.

Definition 5.5. A set $\mathcal{M}$ is a (weak) exponential attractor for the solution semigroup $\{S(t),\ t \geqslant 0\}$ acting in the energy space $\mathcal{E}(p,q)$ if

(1) the set $\mathcal{M}$ is bounded in $\mathcal{E}(p,q)$ and compact in $\mathcal{E}_{-1}$;
(2) it is semi-invariant: $S(t)\mathcal{M} \subset \mathcal{M}$;

(3) it attracts exponentially the images of all bounded in $\mathcal{E}(p,q)$ sets in the topology of $\mathcal{E}_{-1}$, i.e., there exists a positive constant γ and a monotone function Q such that

$$\operatorname{dist}_{\mathcal{E}_{-1}}\big(S(t)B,\mathcal{M}\big)\leqslant Q\big(\|B\|_{\mathcal{E}}\big)e^{-\gamma t}$$

for all bounded subsets $B\subset\mathcal{E}(p,q)$;

(4) it has finite fractal dimension in $\mathcal{E}_{-1}$.

Theorem 5.6. *Let the assumptions of Theorem* 2.2 *hold. Then, the solution semigroup* $\{S(t),\ t\geqslant 0\}$ *associated with Eq.* (1.1) *possesses an exponential attractor* $\mathcal{M}$ *in the sense of Definition* 5.5. *Moreover, this exponential attractor is bounded in* $\mathcal{E}_1$:

$$\|\mathcal{M}\|_{\mathcal{E}_1}\leqslant C.$$

Proof. Indeed, as we have already mentioned in Remark 3.3, the squeezing property (3.4) is sufficient to construct an exponential attractor $\mathcal{M}_d\subset\mathcal{B}_R$ for the discrete semigroup $S(L):\mathcal{E}(p,q)\to\mathcal{E}(p,q)$, see [21]. However, in order to extend it to the attractor $\mathcal{M}$ of the semigroup with continuous time by the standard relation

$$\mathcal{M}=\Big[\bigcup_{t\in[0,L]}S(t)\mathcal{M}_d\Big]_{\mathcal{E}_{-1}},\tag{5.10}$$

we need the uniform Hölder continuity of $S(t)\xi$ on both variables t and ξ on $\mathcal{M}_d$. Although the continuity in ξ follows from Theorem 2.2, we however do not have the Hölder continuity in time if $\mathcal{M}_d$ is only a subset of $\mathcal{E}(p,q)$. So, a priori, more smooth exponential attractor $\mathcal{M}_d$ for the discrete semigroup $S(nL)$ is required.

We overcome this difficulty by considering our solution semigroup in a stronger phase space $\mathcal{E}_1$. Then, due to the dissipative estimate (4.31), we can construct a bounded and invariant absorbing set $\tilde{\mathcal{B}}_R\subset\mathcal{E}_1$ for this semigroup. Furthermore, repeating the arguments given in the proof of Theorem 3.1, but replacing the absorbing set $\mathcal{B}_R\subset\mathcal{E}$ by the new absorbing set $\tilde{\mathcal{B}}_R\subset\mathcal{E}_1$, we construct an exponential attractor $\mathcal{M}_d\subset\tilde{\mathcal{B}}_R\subset\mathcal{E}_1$ for the discrete semigroup $\{S(nL),\ n\in\mathbb{N}\}$ acting in a stronger phase space $\mathcal{E}_1$ (recall that the attraction property is still formulated in the topology of $\mathcal{E}_{-1}$).

Since $\mathcal{M}_d$ is now bounded in the stronger space $\mathcal{E}_1$, a straightforward interpolation gives the desired Hölder continuity

$$\big\|S(t+h)\xi-S(t)\xi\big\|_{\mathcal{E}_{-1}}\leqslant C|h|^{1/2},$$

where the constant C is independent of t and $\xi\in\mathcal{M}_d$. Thus, we can use the standard formula (5.10) in order to construct an exponential attractor $\mathcal{M}\subset\mathcal{E}_1$ for the continuous semigroup $S(t)$ acting in $\mathcal{E}_1$.

So, we have constructed the exponential attractor $\mathcal{M}$ which is bounded in $\mathcal{E}_1$ and attracts bounded *in the stronger space* $\mathcal{E}_1$ sets in the topology of $\mathcal{E}_{-1}$. Finally, in order to verify that this attractor will attract exponentially bounded in $\mathcal{E}(p,q)$ sets as well, it is sufficient to use (5.1) and (2.17) together with the transitivity of exponential attraction, see [26]. Theorem 5.6 is proved. □

Remark 5.7. Theorems 5.1 and 5.6 show that any weak energy solution of Eq. (1.1) possesses an asymptotic smoothing property and belongs to $\mathcal{E}_1$ up to the exponentially decaying terms. However, the dissipativity of, say, *classical* solutions remain an open problem in the 3D case even for globally bounded non-linearity ϕ. Indeed, in order to "freeze" the coefficients and use the maximal regularity of the linearized equation and standard localization technique, we need to control the C^α-norm of $\nabla_x u$ and the $\mathcal{E}_1$-energy gives only $\nabla_x u\in L^6$. By this reason, the additional higher energy estimates are necessary.

The usual way to obtain such estimates (which works perfectly for the case of semi-linear equations) is to multiply the equations by $\partial_t(\Delta_x u)+\varepsilon\Delta_x^2 u$ or something similar. Unfortunately, in the quasi-linear case this procedure produces the additional term like $\phi'''(\nabla_x u)|D_x^2 u|^2 D_x^3 u$ which, in turn, produces the term $\|D_x^2 u\|_{L^4}^4$ (even in the simplest case of periodic BC and bounded ϕ). In the 2D case, this term can be properly estimated by the interpolation

$$\left\|D_x^2 u\right\|_{L^4}^4 \leqslant C\|\Delta_x u\|_{L^2}^2\|\nabla_x\Delta_x u\|_{L^2}^2 \leqslant C\|\xi_u\|_{\mathcal{E}_1}^2\|\xi_u\|_{\mathcal{E}_2}^2$$

(where $\mathcal{E}_2 := W^{3,2}(\Omega)\times W^{2,2}(\Omega)$) and this allows to verify the existence of classical solutions (in fact, even in the case of arbitrary polynomial growth of ϕ; using slightly more accurate estimates, see [24]).

In contrast to that, in the 3D case, we have

$$\left\|D_x^2 u\right\|_{L^4}^4 \leqslant C\|\xi_u\|_{\mathcal{E}_1}^1\|\xi_u\|_{\mathcal{E}_2}^3$$

which is not strong enough to establish the global existence of the $\mathcal{E}_2$-solutions.

We also mention that the problem of further regularity of solutions of (1.1) is closely related with the analogous problem for the simplified pseudo-parabolic equation

$$\gamma\,\partial_t\Delta_x u + \Delta_x u + \nabla_x\big(\phi'(\nabla_x u)\big) = h(t), \qquad u|_{\partial\Omega} = 0 \tag{5.11}$$

for the appropriate external force $h(t)$. However, we do not know how to establish the global existence and dissipativity of more regular (than the H^2-ones) solutions for that equation.

As we have already mentioned in the introduction, there is an alternative way to construct more regular solutions in the particular case of globally bounded ϕ'' based on the integro-differential form (1.12) and perturbation arguments. However, the growing in time bounds for more regular norms seem unavoidable under that method.

We finish this section by considering the semi-linear case $\phi\equiv 0$ (or more general $\phi''(v)\equiv const$). Of course, in that case, the problems mentioned in the previous remark do not appear and the factual regularity of the attractor is restricted only by the regularity of the data (in particular, if Ω, f and g are C^∞, the attractor will belong to C^∞ as well). Moreover, in that case, we are able to obtain slightly more strong result on the attraction property for weak energy solutions, namely, that the constructed global ($\mathcal{A}$) and exponential ($\mathcal{M}$) attractors of the solution semigroup $S(t)$ acting in the energy phase space $\mathcal{E}=\mathcal{E}(q)=\mathcal{E}(1,q)$ attract bounded sets in $\mathcal{E}(q)$ in the topology of the initial energy phase space $\mathcal{E}(q)$ (and not only in a weaker topology of $\mathcal{E}_{-1}$ or $\tilde{\mathcal{E}}$). This can be obtained from the following refining of Theorem 5.1.

Proposition 5.8. *Let the assumptions of Theorem* 2.2 *be satisfied and assume, in addition, that* $\phi\equiv 0$ (*i.e., Eq.* (1.1) *is semi-linear*). *Then the exponential attraction* (5.1) *holds in the topology of the phase space* $\mathcal{E}(q) := [H^1\cap L^{q+2}]\times L^2$, *i.e., there exists a positive constant* γ *and a monotone function* Q *such that*

$$\operatorname{dist}_{\mathcal{E}}\big(S(t)B, \big\{\xi\in\mathcal{E}_1,\ \|\xi\|_{\mathcal{E}_1}\leqslant K\big\}\big) \leqslant Q\big(\|B\|_{\mathcal{E}}\big)e^{-\gamma t} \tag{5.12}$$

for every bounded set B *in* $\mathcal{E}(q)$. *Here* K *is a sufficiently large number and* $\mathcal{E}_1 := H^2\times H^1$.

Proof. Indeed, exactly as in the proof of Theorem 5.1, it is sufficient to verify that

$$\operatorname{dist}_{H^1\cap L^{q+2}}\big(u(t), \big\{u_0\in H^2,\ \|u_0\|_{H^2}\leqslant K\big\}\big) \leqslant C_R e^{-\gamma t} \tag{5.13}$$

for all trajectories u starting from the absorbing set $\mathcal{B}_R$. In turns, in order to prove (5.13), it is sufficient to improve Lemma 5.3 by replacing the space H^1 in estimate (5.8) by $H^1\cap L^{q+2}$. Namely, we only need to prove that

$$\left\|v(t)\right\|_{L^{q+2}}^{q+2} \leqslant C_R\left\|v(0)\right\|_{H^1}^2 e^{-\gamma t}. \tag{5.14}$$

To this end, we first note that, in the semi-linear case, the solution $w(t)$ of problem (5.4) has the additional regularity, namely,

$$\int_t^{t+1} \|\partial_t w(s)\|_{H^2}^2\, ds \leqslant C_R, \tag{5.15}$$

where C_R is independent of t. In order to see that, it is sufficient to express $\partial_t \Delta_x u$ from Eq. (5.4), multiply it by $\partial_t \Delta_x w$ and note that the term $\Delta_x u$ and the non-linearity is under the control due to the embedding $H^2 \subset C$ and estimate (5.6) and the term $h_u(t)$ is controlled by (5.2).

Let us now multiply Eq. (5.5) by $\partial_t v$ and use the following formula

$$\big(f(w+v)-f(w), \partial_t v\big) = \frac{d}{dt}\big[F(v+w)-F(w)-f(w)v\big] - \big(\partial_t w, f(w+v)-f(w)-f'(w)v\big),$$

where $F(u) := \int_0^u f(v)\, dv$. Then, we get

$$\gamma\|\partial_t \nabla_x v\|_{L^2}^2 + \frac{d}{dt}\big[\big(\Phi(v,w),1\big) + 1/2\|\nabla_x v\|_{L^2}^2 + L/2\|v\|_{L^2}^2\big] = \big(\partial_t w, R(v,w)\big), \tag{5.16}$$

where

$$\Phi(v,w) := F(v+w)-F(w)-f(w)v, \qquad R(v,w) := f(v+w)-f(w)-f'(w)v.$$

We now recall the following estimates

$$\begin{cases} 1.\ \big|R(v,w)\big| + \big|\Phi(v,w)\big| \leqslant C|v|^2 + C_1\big[f(v+w)-f(w)\big].v, \\ 2.\ \big|R(v,w)\big| \leqslant C_1|v|^2 + C_2\Phi(v,w) \end{cases} \tag{5.17}$$

for some positive constants C_1 and C_2 which are independent of v and w (these estimates can be easily derived from our assumptions (1.3) on the function f, see [60]). Using estimate (5.17)(2), together with the embedding $H^2 \subset C$, we transform (5.16) as follows:

$$\begin{aligned} &\frac{d}{dt}\big[\big(\Phi(v,w),1\big) + 1/2\|\nabla_x v\|_{L^2}^2 + L/2\|v\|_{L^2}^2\big] + C\|\partial_t w\|_{H^2}\big[\big(\Phi(v,w),1\big) + 1/2\|\nabla_x v\|_{L^2}^2 + L/2\|v\|_{L^2}^2\big] \\ &\quad\leqslant C_1\|\partial_t w\|_{H^2}\|v\|_{H^1}^2. \end{aligned}$$

Finally, multiplying the last inequality by $t-T$, denoting

$$M(v,w) := \big[\big(\Phi(v,w),1\big) + 1/2\|\nabla_x v\|_{L^2}^2 + L/2\|v\|_{L^2}^2\big]$$

(note that $M \geqslant 0$ due to our choice of L) and using (5.17)(1), we have

$$\begin{aligned} &\frac{d}{dt}\big[(t-T)M(v,w)\big] + C\|\partial_t w\|_{H^2}\big[(t-T)M(v,w)\big] \\ &\quad\leqslant C_1(t-T+1)\|\partial_t w\|_{H^2}\|v\|_{H^1}^2 + C_2\big(f(v+w)-f(v), v\big) \end{aligned}$$

which together with the Gronwall inequality gives

$$M\big(v(T+1), w(T+1)\big)$$
$$\leqslant C\|v\|^2_{L^\infty([T,T+1],H^1)} + C\int_T^{T+1} \big|\big(f\big(v(t)+w(t)\big) - f\big(w(t)\big), v(t)\big)\big|\,dt. \tag{5.18}$$

Recall that the first term in the right-hand side of (5.18) decays exponentially due to (5.8) and, in order to see that the second one also decays exponentially, it is sufficient to multiply Eq. (5.5) by v, integrate over $[T, T+1]\times\Omega$ and use (5.5) again. Thus, we have proved that

$$M\big(v(t), w(t)\big) \leqslant C_R \big\|v(0)\big\|^2_{H^1} e^{-\beta t}.$$

Or which is the same (due to (5.8) again),

$$\big(\Phi\big(v(t), w(t)\big), 1\big) \leqslant C_R \big\|v(0)\big\|^2_{H^1} e^{-\beta t}. \tag{5.19}$$

In order to deduce (5.14) from (5.19), it only remains to note that our assumptions on the non-linearity f give the following inequality

$$\Phi(v, w) \geqslant \kappa|v|^{q+2} - C|v|^2$$

for some positive constants κ and C depending on the proper norms of w, see [60]. Thus, estimate (5.14) is verified and Proposition 5.8 is proved. □

Corollary 5.9. *Under the assumptions of Proposition* 5.8*, the global attractor* $\mathcal{A}$ (*resp. the exponential attractor* $\mathcal{M}$) *constructed above, attracts* (*resp. attracts exponentially*) *bounded sets in* $\mathcal{E}(q)$ *in the topology of the space* $\mathcal{E}(q)$ *as well.*

Remark 5.10. To the best of our knowledge, when $\Omega\subset\mathbb{R}^3$ the growth restriction $q\leqslant 4$ has been always posed in order to study the energy solutions. As it follows from our result, this growth restriction is unnecessary and we have well-posedness, dissipativity and asymptotic regularity of weak energy solutions without any restrictions on the growth exponent q.

6. Related equations and concluding remarks

Although only the case of quasi-linear strongly-damped wave equations in the form of (1.1) has been considered in the paper, the developed methods have general nature and can be successfully applied to many related problems. Some of them are briefly discussed below. More detailed exposition of that and similar problems will be given somewhere else.

6.1. Strongly-damped Kirchhoff equation

We start with the so-called Kirchhoff equation with strongly damping term which can be considered as a simplified version of the quasi-linear wave equation (1.1):

$$\begin{cases} \partial_t^2 u - \gamma\partial_t\Delta_x u - \Delta_x u - \Phi\big(\|\nabla_x u\|^2_{L^2}\big)\Delta_x u + f(u) = g, \\ u|_{\partial\Omega} = 0, \quad \xi_u(0) := \big(u(0), \partial_t u(0)\big) = \xi_0, \end{cases} \tag{6.1}$$

where f satisfies (1.3) and $\Phi\in C^1(\mathbb{R}^+\to\mathbb{R}^+)$ satisfies the condition

$$\Phi(s)s - \int_0^s \Phi(\tau)\,d\tau \geqslant 0, \quad \forall s\in\mathbb{R}^+.$$

Indeed, the non-linearity $\Phi(\|\nabla_x u\|^2_{L^2})\Delta_x u = \nabla_x(\Phi(\|\nabla_x u\|^2_{L^2})\nabla_x u)$ is milder than the one $\nabla_x(\phi'(\nabla_x u)\nabla_x u)$ considered before and, by this reason, all the above results can be obtained (analogously, but with great simplifications) for the Kirchhoff equation (6.1) as well. In particular, arguing as in Theorem 2.2, we obtain the uniqueness of energy solutions for any growth exponent q (for the non-linearity f). To the best of our knowledge, this result was known before for $q \leqslant 5$ only.

6.2. Membrane equation

Our next application is the so-called quasi-linear strongly-damped membrane equation which can be considered as a natural 4th order (5th order, being pedantic) analog of problem (1.1):

$$\begin{cases} \partial_t^2 u + \Delta_x \phi(\Delta_x u) + \Delta_x^2 u + \gamma \partial_t \Delta_x^2 u + f(u) = g, \\ u|_{\partial\Omega} = \Delta_x u|_{\partial\Omega} = 0, \quad \xi_u(0) := \big(u(0), \partial_t u(0)\big) = \xi_0, \end{cases} \tag{6.2}$$

where ϕ and f are given non-linearities, $g \in L^2(\Omega)$ is a given source function. We assume that the function $\phi \in C^2$ is monotone and satisfies the analogue of assumptions (1.2)

$$\kappa_2 |u|^{p-1} \leqslant \phi'(u) \leqslant \kappa_1 |u|^{p-1} \tag{6.3}$$

for some positive κ_i and some $p \geqslant 1$ and the non-linearity $f \in C^1$ satisfies the standard dissipativity assumption

$$f(u)u \geqslant -C. \tag{6.4}$$

Analogously the case of Eq. (1.1), it is natural to introduce the weak ($\mathcal{E}$) and the strong energy ($\mathcal{E}_1$) spaces via

$$\mathcal{E} := \big[W^{2,p+1}(\Omega) \cap H^1_0(\Omega)\big] \times L^2(\Omega) \tag{6.5}$$

and

$$\mathcal{E}_1 := \big\{\xi_u \in \big[H^3 \cap W^{2,p+1}(\Omega)\big] \times H^2,\ \phi(\Delta_x u) \in L^2(\Omega)\big\} \equiv \big[H^3 \cap W^{2,2p}(\Omega)\big] \times H^2, \tag{6.6}$$

recall that $H^s := D((-\Delta)^{s/2})$, so the boundary conditions are automatically included. Here we have also used that the condition $\phi(\Delta_x u) \in L^2$ is equivalent to $\Delta_x u \in L^{2p}$.

The energy functional now reads

$$E(\xi_u) := 1/2\|\Delta_x u\|^2_{L^2} + \big(\Phi(\Delta_x u), 1\big) + 1/2\|\partial_t u\|^2_{L^2} + \big(F(u), 1\big) - (g, u),$$

where $\Phi(z) := \int_0^z \phi(v)\,dv$ and $F(z) := \int_0^z f(v)\,dv$ and the standard energy estimate (multiplication of the equation by $\partial_t u + \varepsilon u$) gives

$$\big\|\xi_u(t)\big\|^2_{\mathcal{E}} + \int_0^t \big\|\partial_t u(s)\big\|^2_{H^2}\,ds \leqslant Q\big(\big\|\xi_u(0)\big\|^2_{\mathcal{E}}\big)e^{-\alpha t} + Q\big(\|g\|_{L^2}\big) \tag{6.7}$$

for some positive α and monotone Q.

Thus, in contrast to the case of Eq. (1.1), the C-norm of the solution $u(t)$ is controlled by the energy norm. By this reason, not any additional growth assumption for f is now necessary.

Another essential simplification appears in the proof of the uniqueness of energy solutions. Indeed, following the proof of Theorem 2.2, we now need to multiply the equation for the differences of two solutions by $\Delta_x^{-2}\partial_t v$ (instead of $\partial_t(-\Delta_x)^{-1}v$, $v := u_1 - u_2$) and the only non-trivial term to estimate will be

$$\big(\phi(\Delta_x u_1) - \phi(\Delta_x u_2), (-\Delta_x)^{-1}\partial_t v\big).$$

Since $H^2 \subset C$ and the L^2-norm of the $\partial_t v$ is under the control, this term can be estimated analogously to (2.10) *without* the restrictions on the growth exponent p. Thus, the existence and uniqueness theorem for energy solutions now holds for arbitrary growth of the non-linearity ϕ.

In addition, denoting $v = \partial_t u$, $\mathcal{E}_{-1} := H^{-2} \times H^2$ and repeating word by word the proof of Propositions 2.4 and 2.5, we establish the analogue of estimate (2.19) and the smoothing property:

$$\|\partial_t u(t)\|_{H^2}^2 + \|\partial_t^2 u(t)\|_{H^{-2}}^2 + \int_t^{t+1} \|\partial_t^2 u(s)\|_{L^2}^2\, ds \leqslant \frac{1+t^N}{t^N}\big(Q\big(\|\xi_u(0)\|_{\mathcal{E}}\big)e^{-\alpha t} + Q\big(\|g\|_L^2\big)\big). \tag{6.8}$$

Expressing now the term $\phi(\Delta_x)$ from Eq. (6.2) and using (6.7) and (6.8), we obtain the control of the L^2-norm of $\phi(\Delta_x u)$ or which is equivalent, the control of the $W^{2,2p}$-norm of u. Thus, the semigroup $S(t) : \mathcal{E} \to \mathcal{E}$ associated with problem (6.2) possesses the partial smoothing property of the form

$$S(t) : \big[W^{2,p+1}(\Omega) \cap H_0^1(\Omega)\big] \times L^2(\Omega) \to \big[W^{2,2p}(\Omega) \cap H_0^1(\Omega)\big] \times \big[H^2(\Omega) \times H_0^1(\Omega)\big]. \tag{6.9}$$

The finite-dimensional global attractor $\mathcal{A}$ ($(\mathcal{E}, \mathcal{E}_{-1})$-attractor, to be more precise) can be now obtained repeating word by word the proof of Theorem 3.1. However in contrast to the case of Eq. (1.1), the smoothing (6.9) together with the interpolation

$$\|u\|_{W^{2,p+1}} \leqslant C\|u\|_{W^{2,2}}^{1/(p+1)} \|u\|_{W^{2,2p}}^{p/(p+1)}$$

allows to verify that this attractor is finite-dimensional in $\mathcal{E}$ and attracts in the initial topology of the phase space $\mathcal{E}$ as well, see Remark 3.4. Thus, we have verified the following result.

Theorem 6.1. *Let the above assumptions hold. Then, Eq.* (6.2) *is well-posed and dissipative in the phase space $\mathcal{E}$ and the associated semigroup $S(t)$ possesses a finite-dimensional global attractor $\mathcal{A}$ in $\mathcal{E}$.*

Let us now discuss the further regularity of solutions/attractor of problem (6.2). We first note that the analogue of the main result of Section 4 (strong solutions) is now immediate since the non-linear term arising after the multiplication of (6.2) by $\Delta_x u$ can be estimated via

$$\big(\Delta_x\phi(\Delta_x u), \Delta_x u\big) = -\big(\phi'(\Delta_x u)\nabla_x\Delta_x u, \nabla_x\Delta_x u\big) \leqslant 0$$

and, in contrast to Section 4, we do not have here any problems with boundary terms. By that reason, we need not neither the localization technique of Lemma 4.1 nor the approximations of ϕ by the bounded non-linearities (see the proof of Theorem 4.2) and the existence of a strong solution can be verified by the usual Galerkin method.

Finally, again in contrast to the case of Eq. (1.1), the question about the classical solutions of (6.2) can be solved in a relatively simple way. Indeed, let $\theta := \Delta_x u$. Then this function solves the equation

$$\gamma\partial_t\theta + \theta + \phi(\theta) = h_u(t) := (-\Delta_x)^{-1}\partial_t^2 u + (-\Delta_x)^{-1} f(u) - (-\Delta_x)^{-1} g. \tag{6.10}$$

Since ϕ is monotone and, without loss of generality, $h_u \in L^2([0,T], C^\alpha(\Omega))$ (due to (6.8) and embedding $H^2 \subset C$), then the usual maximum/comparison principle allows to verify the dissipative estimate for θ in $C^\alpha(\Omega)$. Thus, we will have the asymptotic smoothing property to $u(t) \in C^{2+\alpha}(\Omega)$ on the attractor (in fact, in the super-linear case $p > 1$, we even have that $\Delta_x u(t) \in L^\infty(\Omega)$ in a finite time).

As we have already mentioned in Remark 5.7, the regularity $\Delta_x u(t) \in C^\alpha(\Omega)$ is crucial for the application of the standard localization technique and the existence of classical solutions. Thus, we have the following result.

Theorem 6.2. *Let the domain Ω and the data ϕ, f and g be smooth enough and satisfy* (6.3) *and* (6.4)*. Then, the attractor $\mathcal{A}$ of problem* (6.2) *consists of classical solutions. In addition, if the above data is of class C^∞, then the attractor $\mathcal{A}$ belongs to $[C^\infty(\Omega)]^2$ as well.*

To the best of our knowledge, the existence and uniqueness of energy solutions to the problem (6.2) was known only if the non-linearity $\phi(\cdot)$ has a linear growth, see [7] where the abstract differential equation of the form (1.14) is considered (and the assumptions on the linear operators A_1, A_2 and N admit Eq. (6.2) with a sub-linear ϕ as a particular case). The attractor theory for that abstract equation has been developed in [52], but under the additional assumption that $A_2^{-1/2}N$ is a compact operator which automatically *excludes* the case of Eq. (6.2) from consideration (even in the case of sub-linear ϕ). In contrast to that, the methods developed in our paper allow to give the *complete* theory of Eq. (6.2) (existence, uniqueness, regularity, classical solutions, attractors, etc.) for arbitrary growth rate of ϕ.

6.3. The wave equation with structural damping

To continue, we note that our technique allows to improve essentially of the so-called wave equation with structural damping

$$\begin{cases} \partial_t^2 u - \Delta_x u + \gamma(-\Delta_x)^\alpha \partial_t u + f(u) = g, \\ u|_{\partial\Omega} = 0, \quad \xi_u(0) := \big(u(0), \partial_t u(0)\big) = \xi_0, \end{cases} \tag{6.11}$$

where $\alpha \in (1/2, 1]$, $g \in L^2(\Omega)$ and the function f satisfies (1.3) for some exponent q. Indeed, following the scheme of Theorem 2.2, for the uniqueness of energy solutions, we need to multiply the equation for difference between two solutions u_1 and u_2 by $v + \varepsilon(-\Delta_x)^{-\alpha}\partial_t v$ and estimate the term

$$\big(f(u_1) - f(u_2), (-\Delta_x)^{-\alpha}\partial_t v\big) \tag{6.12}$$

using that $(f(u_1) - f(u_2), v) \geqslant -K$ and that the L^2-norm of $\partial_t v$ is under the control. In the case $\alpha > 3/4$, we have the embedding $H^{2\alpha} \subset C$ and the term (6.12) can be estimated exactly as in (2.10). A little more accurate analysis shows that the same estimate still works for $\alpha = 3/4$ without any restrictions on the growth exponent q, but for $\alpha < 3/4$ an additional growth restriction

$$q + 2 < \frac{6}{3 - 4\alpha}, \quad \alpha \in (1/2, 3/4), \tag{6.13}$$

is required (in fact, this technique works also for $\alpha \leqslant 1/2$, but, in this case, the standard scheme of proving the uniqueness and asymptotic regularity is preferable; by this reason, we discuss here the case $\alpha > 1/2$ only).

Thus, analogously to Theorems 2.2 and 3.1, we have the following result.

Theorem 6.3. *Let the assumption* (1.3) *be valid with the exponent q satisfying* (6.13) *(if $\alpha < 3/4$). Then, problem* (6.11) *is well-posed and dissipative in the energy space $\mathcal{E} = \mathcal{E}(q) := [H^1 \cap L^{q+2}] \times L^2$ and possesses the finite-dimensional global attractor $\mathcal{A}$ (in the sense of the definition given in Section* 3 *with $\mathcal{E}_{-1} := H^{1-\alpha} \times H^\alpha$).*

Furthermore, the only multiplication of Eq. (6.11) by $\Delta_x u$ is *insufficient* for proving the existence of strong solutions if $\alpha \neq 1$ since the additional terms $\|\partial_t u\|_{H^1}^2$ and $\partial_t(\nabla_x \partial_t u, \nabla_x u)$ need to be estimated. However, in order to overcome this difficulty, we just need to prove the analogue of Propositions 2.4 and 2.5 (smoothing property for $\partial_t u$) before. Indeed, this smoothing property gives the control of the

$L^2([0,T],H^1)$ and $C([0,T],H^{1-\alpha})$-norms of u and the $C([0,T],H^\alpha)$-norm of $\partial_t u$ and that is *exactly* what we need in order to be able to estimate the above mentioned terms.

Thus, the multiplication by $\Delta_x u$ finally works and we can prove that the attractor $\mathcal{A}$ consists of strong solutions belonging to $\mathcal{E}_1 := H^\alpha \times H^{1+\alpha}$. Since $\alpha > 1/2$, we have $H^{1+\alpha} \subset C$ and the further regularity of the attractor (e.g., classical or C^∞-solution) can be obtained from the linear theory. In addition, the analogue of Proposition 5.8 also can be proved repeating word by word the arguments of Section 5. Thus, the attractor $\mathcal{A}$ is finite-dimensional and attracts the images of bounded sets in the topology of the initial phase space $\mathcal{E}$ as well.

Theorem 6.4. *Let the assumptions of Theorem* 6.3 *hold and let Ω, f and g be smooth enough. Then, the attractor $\mathcal{A}$ consists of classical solutions and attracts the images of bounded sets in $\mathcal{E}(q)$ in the topology of the initial phase space $\mathcal{E}(q)$.*

To the best of our knowledge, the above mentioned results were known before only under the growth restriction $q \leqslant 4$, see [10] (which is usually considered as a critical growth exponent for that equation). In contrast to that, a more or less complete theory for Eq. (6.11) in the case $q \leqslant 4$ (including the limit case $\alpha = 1/2$) can be found in [10–13]. Mention also that, exactly as in the case of Eq. (1.6), the phase space $E(q) = H^1_0 \times L^2$ becomes independent of q if $q \leqslant 4$ and the assumptions on the non-linearity f can be slightly weaken, see (1.9) and (1.10).

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